Algebra in Problem Solving (Senior)

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March 29, 2016

1 Polynomials

Definition. A polynomial is an expression of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0.$$

- n is the *degree* of the polynomial.
- $a_n, a_{n-1}, ..., a_1, a_0$ are the *coefficients* of the polynomial.
- a_n is the *leading coefficient*.
- The polynomial is *monic* if the leading coefficient is 1.
- A *root* to the polynomial is a solution to the equation $P(x) = 0$.

1.1 Quadratics

Quadratics are polynomials of degree 2. Given a quadratic

$$ax^2 + bx + c$$

its roots are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The discriminant is

$$\Delta = b^2 - 4ac.$$

- If $\Delta > 0$, we have 2 distinct real roots.
- If $\Delta = 0$, we have exactly one real root.
- If $\Delta < 0$, we have no real roots.

1.2 Polynomial Operations

We can add and subtract polynomials by collecting like terms. We can multiply polynomials by multiplying each term of one with each term of the other. But can we divide polynomials? It may not be obvious but we can. To divide $P(x)$ by $Q(x)$, we subtract multiples of $Q(x)$ for $P(x)$ until the remainder is small. We will define small to mean that the degree of the remainder must be less than the degree of $Q(x)$. Hence, if we're dividing by a linear polynomial, the remainder will be constant, if we're dividing by a quadratic, the remainder will be at most linear. If the remainder is zero, then we say that $Q(x)$ *divides* $P(x)$, of $Q(x)$ is a *factor* of $P(x)$.

1.3 Remainder Theorem

Let $P(x)$ be a polynomial and a is a constant. If we divide $P(x)$ by $(x - a)$ then the remainder has degree < 1 so is a constant. So we can write

$$P(x) = (x - a)Q(x) + r.$$

Substituting $x = a$ gives $P(a) = r$ and yields the remainder theorem.

Remainder Theorem

If $P(x)$ is divided by $(x - a)$ then the remainder is $P(a)$.

Factor Theorem

If $P(a) = 0$ then $(x - a)$ divides $P(x)$.

1.4 Viete's Formula

Consider a monic polynomial of degree 2

$$P(x) = x^2 + bx + c.$$

Let its roots be α and β, where $\alpha, \beta \in \mathbb{C}$. The roots are guaranteed to exist by the quadratic formula. Hence by the factor theorem, $P(x) = a(x - \alpha)(x - \beta)$, and since our polynomial is monic, $a = 1$. So $P(x) = x^2 - (\alpha + \beta)x + \alpha\beta$, and thus

$$\begin{aligned} b &= -(\alpha + \beta) \\ c &= (\alpha\beta) \end{aligned}$$

Hence the sum of the roots of $x^2 + bx + c$ is $-b$ and the product of the roots is c.

Doing a similar thing with a monic cubic $P(x) = x^3 + bx^2 + cx + d$ with roots α, β and γ gives us the following relations:

$$\begin{aligned} b &= -(\alpha + \beta + \gamma) \\ c &= (\alpha\beta + \alpha\gamma + \beta\gamma) \\ d &= -(\alpha\beta\gamma) \end{aligned}$$

1.5 Types of polynomials

Lets look at the definition of a polynomial again.

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0.$$

We haven't defined what the coefficients $a_n, a_{n-1}, ..., a_1, a_0$ can be. If they all belong to some set S, then we say that $P(x)$ is a polynomial over S. For example, if the coefficients are all integers, then $P(x)$ is a polynomial over the integers, or an integer polynomial. If the coefficients are all reals, then $P(x)$ is a polynomial over the reals, or a real polynomial.

1.6 Roots

The fundamental theorem of algebra states that every non-constant polynomial over the complex numbers has a complex root. We will not prove this, it is extremely difficult. As a corollary of the fundamental theorem of algebra, a degree n polynomial has exactly n complex roots. This results in the following useful lemma:

Lemma

A real polynomial can always be factorised into real quadratic and linear factors.

Lemma

A polynomial with infinitely many roots is the zero polynomial.

Problems

1.7 Easy

1. Which of the following expressions are polynomials?
$$4x^2 + x - 2,\ x^3 + 2x^2 - x - 1,\ \sqrt{x},\ 1,\ \frac{1}{x}.$$

2. Graph the following polynomials
   (a) $x^2 + 4x + 4$
   (b) $(x+1)(x-1)(x-3)$

3. The polynomial $a^2x^2 + (2a+1)x + 2$ has exactly one real root. Find a.

4. $\frac{x^2+3x+2}{2x+5} = k$. For which real k does this equation have at least one real solution for x?

5. What is $(x^2 - 4x + 3) \times (x+2)$?

6. Divide $x^5 + 3x^3 + 2x^2 + x + 1$ by $x^2 + x + 1$.

7. What is the highest common factor of $x^4 + x^3 - x^2 + 1$ and $x^3 + 2x^2 + 2x + 1$.

8. Is $x^2 - 6x + 8$ a multiple of $(x-2)$?

9. Is $3x^3 + 2x^2 - x + 1$ a multiple of $(x-2)$?

10. Solve $x^3 - 7x^2 + 6 = 0$.

11. Prove that $a - b \mid P(a) - P(b)$ for any polynomial $P(x)$.

1.8 Hard

1. Let $P(x)$ be a polynomial such that $P(1) = 1$ and $P(2) = 7$. Find the remainder when $P(x)$ is divided by $x^2 - 3x + 2$.

2. $x^3 + 3x^2 + 4x + 5$ has roots α, β, γ.
   (a) Find $\alpha^2 + \beta^2 + \gamma^2$.
   (b) Find $\alpha^3 + \beta^3 + \gamma^3$.

3. Find Viete formulations for quartics and higher degree polynomials.

4. Which quadratic polynomials have linear real factors, but no linear rational factors?

5. Find all real polynomials $P(x)$ such that $P(n) = P(n^2)$ for all $n \in \mathbb{N}$.

6. Find all real polynomials $P(x)$ such that $P(q) + 1 = P(q+1)$ for all primes $q \in \mathbb{N}$.

7. *(Eisenstein's Criterion)* Suppose we have an integer polynomial $P(x) = a_nx^n + ... + a_1x + a_0$ and there is a prime p such that p is a divisor of every coefficients except a_n, and p^2 is not a divisor of a_0. Then $P(x)$ is irreducible over the integers.

8. *(Rational Root Theorem)* If the rational polynomial $P(x) = a_nx^n + a_{n-1} + ... + a_2x^2 + a_1x + a_0$ has root $\frac{p}{q}$ in simplest form, then p divides into a_0, and q divides into a_n.

9. Prove that $(x^a - 1)$ divides $(x^b - 1)$ iff $a|b$.

10. Let $P(x) = x^3 - 4x^2 + 11x - 7$ have zeroes α, β and γ. Find the monic cubic polynomials $Q(x)$ and $R(x)$ whose zeroes are $\alpha^2, \beta^2, \gamma^2$ and $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ respectively.

11. Find all polynomials such that $P(x)^2 = P(x^2)$.

2 Functions and Graphs

Often students are taught that functions are like vending machines. Essentially you put in a number, some cogs turn and the machine returns a (possibly different) number. Feel free to use this analogy. I prefer to think of a function as a pairing of numbers, where the first number in each pair is what goes into the function and the second number is what comes out. A more technical term for these pairs is a *relation* with one right number for each left number. We write $f(x)$ and say 'f of x' .
Many common functions include:

- Polynomial : $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$
- Square Root : $f(x) = \sqrt{x}$
- Exponential : $f(x) = e^x$
- Power : $f(x) = x^c$
- Logarithm : $f(x) = \ln(x)$
- Trigonometric : $f(x) = \sin(x)$ etc.
- Reciprocal : $f(x) = \frac{1}{x}$
- Floor : $f(x) = \lfloor x \rfloor =$ greatest integer less than or equal to x
- Fractional : $f(x) = \{x\} = x - \lfloor x \rfloor$
- Ceiling : $f(x) = \lceil x \rceil =$ smallest integer greater than or equal to x
- Absolute Value : $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

When discussing functions, it really is important to specify what kind of numbers can be put into a function. Sometimes you want to restrict what goes into the function for a particular reason. To be able to do this, let's have a quick recap of what sets are.

2.1 Sets

A set is a collection of elements without repetition. Here are a few examples of sets:

- $\mathbb{N} = \{1, 2, 3, \dots\} =$ the natural numbers or positive integers.
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\} =$ the integer numbers or just integers.
- $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}\} =$ the rational numbers. Also known as fractions (ratios of integers).
- $\mathbb{R} =$ the real numbers. All of the numbers you would find on a number line.
- $\mathbb{C} =$ the complex numbers.

There is a little bit of extra notation to understand about sets. For the explanations below assume $A = \{\text{cat}, \text{dog}\}$ and $B = \{\text{cat}\}$.

- When we write out the elements of a set, we put them inside curly braces, eg. $A = \{\text{cat}, \text{dog}\}$.
- $a \in A$ means a is an *element* of A. For example, cat $\in A$.
- $B \subset A$ means B is a *subset* of A. In our case, $B \subset A$ or $\{\text{cat}\} \subset A$.
- $A \cap B$ is the *intersection* of A and B, which is the common elements to both sets. In our case $A \cap B = \{\text{cat}\}$.

- $A \cup B$ is the *union* of A and B, which is all the elements in either set. In our case $A \cup B = \{\text{cat}, \text{dog}\}$.
- A^c is the *complement* of A, which is all the elements not in A. Here it is really important to know what universal set we are using. If for example our setting is the set of all pets, then A^c is the set of all pets that are not dogs or cats.

We will also sometimes add little symbols to well-known sets. For example,

- $\mathbb{R}^+$ is the set of all positive real numbers.
- $\mathbb{N}_0$ is the set of all natural numbers and zero.
- $\mathbb{Q}_0^-$ is the set of all non-positive rational numbers.

When working with the reals, we sometimes like to talk about intervals. Intervals are all the reals between two numbers. If we are talking about real numbers between a and b, we write it as:

- $(a, b) =$ interval between a and b not including a and b.
- $(a, b] =$ interval between a and b not including a and including b.
- $[a, b) =$ interval between a and b including a and not including b.
- $[a, b] =$ interval between a and b including a and b.

2.2 Properties of Functions

Now that we are more familiar with functions, I would like to be a bit more technical. Functions are dependent on two very important sets: the *domain* and *codomain*. The domain is the set of all numbers (things) we can put into the function and the codomain is the set of all numbers (things) we are allowed to get out. If A is the domain and B is the codomain, we write $f : A \to B$.
The *range* is the set of all numbers that actually come out of the function. By definition, range $\subset$ codomain.
A function where every x value gives a unique y value is an *injective* (or *one-to-one*) function. In particular, if $f(x) = f(y)$ then $x = y$. A function that is not injective is also called many-to-one. A function where you can get every value in the codomain is *surjective* (or *onto*). In particular, there is a solution for the equation $f(x) = c$ for all $c \in$ codomain. A function that is both injective and surjective is called *bijective*.
We can *compose* functions, which is an interesting way of putting two functions together. Imagine taking x and putting it into f and then the result of that into g. That is called $g(f(x))$ or $(g \circ f)(x)$. When $g(f(x)) = f(g(x)) = x$, then we say that g is the *inverse* of $f : A \to B$ and write $g = f^{-1} : B \to A$.
There are other cool properties of certain functions:

- Even. $f(-x) = f(x)$ for all x.
- Odd. $f(-x) = -f(x)$ for all x.
- Periodic. $f(x + p) = f(x)$ for all x.
- Involution. $f(f(x)) = x$ for all x.
- Fixed point. These are points where $f(x) = x$.

2.3 Graphs

The thing that you draw with an x-axis and a y-axis is called the Cartesian plane. It is the 2D version of the real number line. It is supposed to be infinitely long in each direction, but usually we focus on a particular area of the Cartesian plane (and this is usually near the origin - the point where the axes meet). Points in this plane are of the form (x, y) where x and y are real numbers. Keeping this in mind, the graph of a function of a function $f(x)$ is $(x, f(x))$. We can write this in a more familiar form:

$$y = f(x).$$

Notice that for a function there is only one $f(x)$-value for every x, so on the graph there will only be one y-value for every x.

2.4 Shape of Graphs

Let's talk about the shape of functions for a little while. An *increasing* function f is such that $f(x) \geq f(y)$ whenever $x > y$. A *strictly increasing* function f is such that $f(x) > f(y)$ whenever $x > y$. Similarly, one can define *decreasing* and *strictly decreasing* functions. A function that is either increasing or decreasing is called *monotonic*.

Another very important concept is continuity. We say a function is *continuous* if we can draw its graph without taking the pen off the paper. More technically f is continuous at a if $\lim_{x \to a} f(x) = f(a)$. But we aren't going to worry about technicalities. Most of the functions you know are continuous. If a function is continuous we can do this awesome thing called:

Intermediate Value Theorem. *Suppose f is a continuous function defined on $[a_1, a_2]$ and $f(a_1) = b_1$ and $f(a_2) = b_2$. Then for any $b \in [b_1, b_2]$, there exists $x \in [a, b]$ such that $f(x) = b$.*

Think about what this means. It is saying that if the function is continuous and at some point it is equal to b_1 and somewhere it is equal to b_2, then somewhere in between it has to equal all of the numbers between b_1 and b_2. Amazing!

A function is *concave* if it looks like a sad face. If you draw a chord it should be completely below the graph. A *convex* function is like a happy face and the chord is completely above the graph. This can have implications for finding maxima and minima of functions.

The following are good techniques to consider when working with graphs:

- Combine graphs: add, subtract.
- Use properties of the common functions.
- Find roots (x-intercepts), y-intercepts, asymptotes.
- Figure out where it is positive, negative.
- Figure out where it is increasing, decreasing, concave, convex.
- Figure out where it is injective, surjective, bijective.
- Look out for other properties of functions: even, odd, periodic, involution, fixed point.
- Look for continuity and the Intermediate Value Theorem.

2.5 Quadratics

Quadratic functions are polynomials as we have already seen, so you should have a good understanding of them. The graph of a quadratic is called a parabola. It looks like the trajectory of a ball when you throw it. In fact, one cool property of it is that a parabola is the shape you get if you want to find all the points that are equally distant from a specific point and a specific line. When working with quadratics, it is important to be able to find the roots and the vertex. You can find the equation of the roots using the quadratic formula. If the parabola is

$$y = ax^2 + bx + c$$

then the vertex will be the point

$$\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right).$$

Problems

2.6 Easy

1. What is the range of $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$? What if $f : \mathbb{N} \to \mathbb{N}$?
2. What is a graphical test for injectivity?
3. Find two examples of injective, but not surjective functions. Find two examples of surjective, but not injective functions.
4. If $f(g(x)) = \frac{5}{\sqrt{3^{x+1}}}$, what are the possibilities for f and g?
5. What property must f have to have an inverse? Why is the inverse of a function, the reflection of the function in $y = x$?
6. Do involutions have inverses and what are they?
7. Show that $P(x) = 2x^3 - 8x^2 + 3x - 2$ has at least one root. Give an interval of length at most 1 in which such a root lies.
8. Consider how the graph transforms if we go from $y = f(x)$ to
   - $y = af(x) + d$,
   - $y = f(bx + c)$,
   - $y = af(bx + c) + d$.

   Try this for a few of the types of functions from the previous question. Focus on what happens graphically.
9. Prove that a strictly monotonic function is injective.
10. Give an example of a non-continuous function.

2.7 Hard

1. What functions are periodic and monotonic?
2. Determine all increasing involutions.
3. Find all solutions $x \geq 1$ to the equation $x^3 = \lfloor x \rfloor^3 + \{x\}^3$.
4. Prove that there are two points on Earth which have the exact same temperature? [Hint consider points opposite each other]
5. Prove that you can always balance a rectangular table with four legs at a specific on uneven ground by simply rotating it.
6. If $f : \mathbb{Z} \to \mathbb{Z}$ and $f(x + y) = f(x) + f(y)$, what is $f(n)$ in terms of $f(1)$?

3 Functional Equations

Functional equations will probably be a very new type of problem from anything you have used before. Usually the problem will be to find all functions that satisfy an equation that involves a function. For example,

Motivating Problem. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ with the property:

$$f(x + f(y)) = f(x) + y$$

for all rational numbers x, y.

There is no algebra to do and there is no graph to draw, because we don't know what $f(x)$ is. You may also ask about the y. Why is there an x and a y? We will discuss this by looking at an example. Consider trying $f(x) = x^2$. Essentially, what you put into f gets squared. So, $f(y) = y^2$. Thus,

$$LHS = f(x + f(y)) = f(x + y^2) = (x + y^2)^2 = x^2 + 2xy^2 + y^4. \text{ Meanwhile } RHS = f(x) + y = x^2 + y.$$

So, $f(x) = x^2$ is a solution if and only if the $LHS = RHS$ for all $x, y \in \mathbb{Q}$. This means $x^2 + 2xy^2 + y^4 = x^2 + y$ for all $x, y \in \mathbb{Q}$. But this is false. You can tell it is false because the two expressions don't look the same, so I'm sure I can find x and y that make it false. Indeed $x = 0, y = 2$ has the $LHS = 16$, while the $RHS = 2$. So, $f(x) = x^2$ is *not* a solution.
Turns out $f(x) = x$ for all $x \in \mathbb{Q}$ works as does $f(x) = -x$ for all $x \in \mathbb{Q}$. Please check this by substituting it in. It turns out these are *all* the solutions. No other functions work! You should be a little shocked because there are an infinite number of functions for each of an infinite number of types of functions. The amount of possible functions is mindboggling and for these to be the only solutions!?

3.1 The Basics

You cannot just substitute in types of functions. It would be like substituting numbers for x when $x \in \mathbb{R}$. There is an infinite number, so it cannot be done that way. Here are a few important points to know.

- A function $f : X \to Y$ sends each value in X to a unique value in Y.
- Substitutions are where you set x or y to a specific value. For example, setting $x = 0$ in Problem 1 above yields:
$$f(f(y)) = f(0) + y$$
whereas setting $y = 0$ yields:
$$f(x + f(0)) = f(x).$$
- A function f is *injective* if no two numbers in X are sent to the same value in Y. Mathematically, one proves this by starting with $f(a) = f(b)$ and showing that this implies $a = b$.
- A function f is *surjective* if the function achieves every value in Y. Mathematically, one proves this by taking a random number $c \in Y$ and finding a $k \in X$, such that $f(k) = c$.
- Other concepts worth knowing: domain, co-domain, range, monotone, continuous, bijective, inverse function, fixed points, involutions. Consult the functions handout for more information about these.

3.2 Practice

For most functional equation problems, injectivity and surjectivity are incredibly vital tools. For this reason, before this handout moves on to other concepts, the author recommends the reader try these problems.

1. $f : \mathbb{Q} \to \mathbb{Q}$ satisfies $f(x + y) = f(x) + f(y) + 1$. Find $f(0)$.
2. $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(1 - x) = xf(x) - 2x + 1$. Find $f(0)$.

3. $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y) = f(x)f(y)$. Find all possible values for $f(0)$, and prove that $f(x) = 0$ in the case where $f(0) = 0$.

4. $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(f(x)) = x$. Prove that f is injective.

5. $f : \mathbb{Q} \to \mathbb{Q}$ satisfies $f(x + f(y)) = f(x) + y$. Prove that f is injective.

6. $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(f(x)) = x$. Prove that f is surjective.

7. $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(xf(x) + f(y)) = (f(x))^2 + y$. Prove that f is surjective.

3.3 Techniques

We solve functional equation problems by finding more and more about the function. For example, we may try to find specific values of the function or properties such as injectivity or surjectivity. Here is a list of techniques that tends to yield results.

- Substitute in simple values: $x = -1, 0, 1, 2$
- Substitute in simple values: $x = y$, $x = -y$
- Substitute in values to make things cancel.
- Prove that your function is injective, and use this fact to "cancel f's". Note that $x = f(a)$ may be useful for injective functions.
- Prove that your function is surjective. Hence let k be a number such that $f(k) = 0$ (or some other number) and set a variable to be k.
- Reduce the problem to Cauchy's Functional Equation:
  - If $f : \mathbb{Q} \to \mathbb{Q}$ satisfies $f(x+y) = f(x) + f(y)$ then $f(x) = cx$ for some constant c.
  - If $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y) = f(x) + f(y)$ and f is continuous or monotone, then $f(x) = cx$ for some constant c.
- Other things to try: Make the function simpler via transformations, make one side of the equation look like the other side, consider fixed points, look for symmetry or breaks in symmetry, use induction or the Fundamental Theorem of Algebra for integer functions.

3.4 Back to the motivating problem

Recall the motivating problem:

Motivating Problem. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ with the property:

$$f(x + f(y)) = f(x) + y$$

for all rational numbers x, y.

Proof. 1. *We first show f is injective.* Suppose $f(a) = f(b)$. Letting $y = a$ and then $y = b$ yields $f(x + f(a)) = f(x) + a$ and $f(x + f(b)) = f(x) + b$. Given that $f(a) = f(b)$, we find

$$f(x + f(a)) = f(x + f(b)) \qquad \Rightarrow f(x) + a = f(x) + b \qquad \Rightarrow a = b$$

as required.

2. *We next find* $f(0)$. Set $y = 0$. Then

$$f(x + f(0)) = f(x) \text{ and because } f \text{ is injective } \Rightarrow x + f(0) = x \qquad \Rightarrow f(0) = 0.$$

3. *We now show f is an involution.* Set $x = 0$. Given that $f(0) = 0$, we get

$$f(f(y)) = y.$$

4. *We now show f satisfies Cauchy's Functional Equation.* As f achieves values in $\mathbb{Q}$, it is fine to let x or y equal to values of the function. In particular, set $y = f(z)$ for $z \in \mathbb{Q}$. Then

$$f(x + f(f(z))) = f(x) + f(z) \qquad \Rightarrow f(x+z) = f(x) + f(z)$$

using the fact that it's an involution. So, $f(x) = xc$ for some $c \in \mathbb{Q}$.

5. *Check the solution!* You will lose marks if you don't substitute your solutions back into the functional equation. This is particularly important here, because $f(x) = xc$ isn't right. Only certain values of c work. When we substitute it in we get:

$$\begin{aligned} f(x + f(y)) = f(x) + y \qquad &\Rightarrow f(x + cy) = cx + y \\ &\Rightarrow c(x + cy) = cx + y \\ &\Rightarrow cx + c^2 y = cx + y \\ &\Rightarrow c^2 = 1 \end{aligned}$$

giving $c = \pm 1$. Trying $f(x) = x$ and $f(x) = -x$ yields identical left and right hand sides so they work.

6. *State the answers* The solutions are

$$\boxed{f(x) = x \quad \forall x \in \mathbb{Q}} \qquad \text{or} \qquad \boxed{f(x) = -x \quad \forall x \in \mathbb{Q}}.$$

□

3.5 Cauchy's Functional Equation

This is such an important and instructive functional equation that I will provide a solution to it and of course the accompanying proof. So we state the equation and solution first:

Cauchy's Functional Equation. The only functions $f : \mathbb{Q} \to \mathbb{Q}$ that satisfy

$$f(x + y) = f(x) + f(y)$$

are of the form $f(x) = ax$ for some constant $a \in \mathbb{Q}$.

Proof. 1. *First we show $f(0) = 0$.* Letting $x = y = 0$ yields $f(0) = 2f(0)$, so $f(0) = 0$.

2. *Using induction, we show that $f(nx) = nf(x)$ for $n \in \mathbb{N}$.* This is obviously true for $n = 1$, so we consider this to be the base case for the induction. Now suppose it is true for $n = k \in \mathbb{N}$. That is, $f(kx) = kf(x)$. Then set $x = kx$, $y = x$. This yields

$$f(kx + x) = f(kx) + f(x) \qquad \Rightarrow f((k+1)x) = kf(x) + f(x) = (k+1)f(x)$$

as required completing the induction.

3. *We show that $f(n) = nf(1)$ for $n \in \mathbb{N}$.* This follows from the previous statement, by setting $x = 1$. Thus,

$$f(n) = nf(1) \text{ for } n \in \mathbb{N}.$$

4. *We show that $f(k) = kf(1)$ for $k \in \mathbb{Q}^+$.* Take the result from point 2 and substituting $x = \frac{m}{n}$ yields $f(m) = nf\left(\frac{m}{n}\right)$. So,

$$f\left(\frac{m}{n}\right) = \frac{1}{n} f(m) = \frac{m}{n} f(1).$$

Thus this result follows.

5. *We show that $f(x) = cx$ for some $c \in \mathbb{Q}$ and all $x \in \mathbb{Q}$.* Let $f(1) = c$. Thus, we currently have $f(k) = kc$ for $k \in \mathbb{Q}_0^+$ if we combine the results of point 1 and 4. Now, set $y = -x$ into the original equation. This yields $f(0) = f(x) + f(-x)$. Thus, $f(-x) = -f(x)$, so f is odd. In particular, for $k \in \mathbb{Q}^+$,

$$f(-k) = -f(k) = (-k)c.$$

   Thus,

$$f(k) = kc \quad \forall x \in \mathbb{Q}.$$

6. *We need to check now. $LHS = f(x+y) = (x+y)c = xc + yc = f(x) + f(y) = RHS$* as required.

$\square$

It is up to the reader to try and prove that monotonicity, multiplicativity or continuity allows one to extend this result to all of $\mathbb{R}$. But be careful! There are many functions that satisfy Cauchy's equation that over the reals do not satisfy this nice form.

Problems

3.6 Easy

1. Find all $f : \mathbb{Q} \to \mathbb{Q}$ satisfying:
$$f(x+y) = f(x) + f(y) + 1.$$
2. Find all $f : \mathbb{R} \to \mathbb{R}$ satisfying:
$$f(1-x) = xf(x) - 2x + 1.$$
3. Find all $f : \mathbb{R} \to \mathbb{R}$ satisfying:
$$f(x - f(y)) = 1 - x - y.$$
4. Find all $f : \mathbb{Q} \to \mathbb{Q}$ satisfying:
$$f(x+y) = f(x) + f(y) + 2xy.$$
5. Find all $f : \mathbb{R} \to \mathbb{R}$ satisfying:
$$f(x^2 + y) = 2^y f(x - y).$$
6. Find all $f : \mathbb{R} \to \mathbb{R}$ such that
$$xf(y) - yf(x) = (x - 2y)(f(x) + 1)f(y).$$
7. Find all $f : \mathbb{N} \to \mathbb{N}$ satisfying:
   (a) $f(f(n) + m) = n + m$
   (b) $f(45) = 79$

3.7 Hard

1. Find all $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(f(x-y)) = f(x) - f(y) + f(x)f(y) - xy.$$
2. Find all $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2 + f(y)) = (x-y)^2 f(x+y).$$
3. Find all $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^2 - y^2) = xf(x) - yf(y).$$
4. Let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be such that $f(n+1) > f(n)$ and $f(n + f(m)) = f(n) + m + 1$. Find all possible values of $f(2001)$.
5. Find all $f : \mathbb{R} \to \mathbb{R}$ satisfying:
$$f(xf(x) + f(y)) = (f(x))^2 + y.$$

4 Inequalities

At school, we mostly work with equations. These are two expressions separated by the equality symbol, =, called 'equals'. However, in mathematics it often pays to talk about inequations. If we want to say two things are unequal we can stick an inequality sign, $\neq$, between them. We can be more precise by saying which side is bigger by using the following symbols:

- $<$ is less than
- $>$ is more than
- $\leq$ is less than or equal to
- $\geq$ is more than or equal to

All inequalities will involve these four symbol $>$, $<$, $\geq$ or $\leq$. Here they are in action:

$$5 > 3, \quad 2 < 7, \quad 8 \geq 1, \quad 4 \leq 3 + 1.$$

Note, that it is ok to say $8 \geq 1$ because even though 8 is more than 1 it is also more than or equal to 1.

4.1 Solving Inequalities

Here are the things you can do with inequalities:

- Add or subtract things from both sides of an inequality. So, if I start with $2x + 1 > 5$, I can happily get: $2x > 4$ (subtracting 1 from both sides), or $-4 > -2x$ (subtracting $2x + 5$ from both sides).
- Multiply or divide both sides of an inequality. CAUTION: if you multiply or divide by a negative value, the inequality sign flips (from $<$ to $>$ and vice versa or from $\leq$ to $\geq$ or vice versa).
- You have to be careful with other algebraic manipulations. In general, you can't square both sides. The reason is that on one side you might be squaring a negative number (which should flip the inequality), but on the other side you might be squaring a positive number (which does not flip the inequality)! The rule of thumb is that if both sides are the same sign (positive or negative) you can square. Just remember that if both sides were negative do not forget to flip the inequality.
- Other algebraic manipulations such as exponentiation or logarithms you can do, but many others like sine or cosine you cannot. The general rule is if a function is completely increasing you can apply it and if a function is completely decreasing you can apply it, but you have to flip the inequality.

I now provide a few examples of simple solving of inequalities:

Problem 1. Solve $2x + 1 < 5$.

Proof. Subtracting 1 from both sides yields $2x < 4$. Dividing both sides and remembering that we do not flip the inequality because 2 is positive, gives

$$x < 2.$$

This is the answer. All numbers x which are less than 2 satisfy the inequality. □

Problem 2. Solve $\frac{x+1}{x-1} \leq 2$.

Proof. Note, the inequality is $\leq$ not $<$, so do not forget to keep using the correct symbol. The first thing we would like to do is multiply by $x - 1$, but we need to be careful. What if it is negative? So, we have two cases:

Case 1: $x - 1 < 0$. This is the same as saying $x < 1$. Now, when we multiply both sides by $x - 1$ we have to flip the inequality so we get

$$x + 1 \geq 2x - 2.$$

Combining a few steps into one, I now subtract x from both sides and add 2 to both sides giving me

$$3 \geq x \text{ or equivalently } x \leq 3.$$

So, in this case where $x < 1$ we get that $x \leq 3$, which together implies all x that are less than 1 ($x < 1$).
Case 2: $x - 1 \geq 0$. This is the same as saying $x \geq 1$. Now when we multiply both sides by $x - 1$ we leave the inequality sign so we get

$$x + 1 \leq 2x - 2.$$

Again applying standard algebraic techniques we get

$$3 \leq x \text{ or equivalently } x \geq 3.$$

So, in this case where $x \geq 1$, we get $x \geq 3$, so together this implies $x \geq 3$.
Thus, the answers for this inequality are $x < 1$ and $x \geq 3$. □

4.2 The Strongest Inequality

From here on in, we will no longer concentrate on solving inequalities. In much the same way that in school maths, you have finished learning how to solve maths equations, after a little bit of practice, inequalities should be easy to solve too.
What we do now is prove that certain inequality statements are always true for all values of our variables. A lot of this is based on one very simple inequality that we all know and that is especially strong:

Theorem 4.1.

$$x^2 \geq 0 \qquad \forall x \in \mathbb{R}.$$

This says that all real numbers squared are nonnegative (positive or zero). Pretty straightforward.
Let's have a look at how powerful this is.

Problem 3. Prove that for all real numbers a and b,

$$a^2 + b^2 \geq 2ab.$$

Proof. This should be shocking. Why should the left hand side always be bigger than the right hand side? Well here is why. We know from our strongest inequality that any real number squared is nonnegative. Therefore,

$$(a - b)^2 \geq 0.$$

But this expands to $a^2 - 2ab + b^2 \geq 0$, which when we rearrange becomes

$$a^2 + b^2 \geq 2ab.$$

□

4.3 Useful Inequalities

There are many very useful inequalities that have been proved for us using complicated techniques.

Theorem 4.2. *(Discrete Inequality) If $a, b \in \mathbb{Z}$ and $a > b$, then $a \geq b + 1$.*

This is easy to understand. If your variables are whole numbers and one is bigger than the other, then it must be at least one bigger than the other!

Theorem 4.3. *(Rearrangement Inequality) Let $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$ be real numbers and let $z_1, z_2, \ldots, z_n$ be any permutation of $y_1, y_2, \ldots, y_n$. Then*

$$x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1 \leq x_1 z_1 + x_2 z_2 + \cdots x_n z_n \leq x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Again, this one can be understood in a common sense way. It is easiest to understand with an example scenario. Someone is offering you money. You are shown three piles of money which contain \$10 notes, \$20 notes and \$50 notes respectively. You are told that you are allowed to take 1 note from one pile, 2 notes from another pile and 5 notes from another pile. Obviously, the worst (smallest) choice is to take 1 \$50 note, 2 \$20 notes and 5 \$10 notes, while the best option is to take 5 \$50 notes, 2 \$20 notes and 1 \$10 note.
What we've got are two sequences: \$10, \$20, \$50 and 1, 2 and 5. If we pair them up in order we get the biggest return. If we pair them up in reverse order we get the worst result. Whereas any other pairing is in between. This is the rearrangement inequality.

Theorem 4.4. *(AM-GM-HM Inequality) For positive real numbers $x_1, x_2, \ldots, x_n$,*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

with equality iff $x_1 = x_2 = \cdots = x_n$.

Theorem 4.5. *(Cauchy-Schwarz Inequality) For* all *real numbers $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$,*

$$(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2 \leq (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2)$$

Equality holds iff $y_1 = rx_1$, $y_2 = rx_2, \ldots, y_n = rx_n$ for some real constant r.
There is an alternative version that is perhaps easier to use, but it works only for positive *reals $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$:*

$$(\sqrt{x_1y_1} + \sqrt{x_2y_2} + \cdots + \sqrt{x_ny_n})^2 \leq (x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n)$$

Theorem 4.6. *(Jensen's Inequality) If $f : [a, b] \mapsto \mathbb{R}$ is convex, then*

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \geq f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)$$

Note that if f is concave, then the inequality points the other way.

Theorem 4.7. *(Power Mean's Inequality) This is a generalisation of the AM-GM-HM inequality. The* power mean *of order r is defined as*

$$P(r) = \left(\frac{x_1^r + x_2^r + \cdots + x_n^r}{n}\right)^{\frac{1}{r}}$$

Then $r < s \Rightarrow P(r) \leq P(s)$. Note, $P(1) = AM$, $P(0) = GM$, $P(-1) = HM$.

Theorem 4.8. *(Weighted Power Mean's Inequality) The* weighted power mean *of order r is defined as*

$$P(r) = \left(\frac{\omega_1 x_1^r + \omega_2 x_2^r + \cdots + \omega_n x_n^r}{\omega_1 + \omega_2 + \cdots + \omega_n}\right)^{\frac{1}{r}}$$

Then $r < s \Rightarrow P(r) \leq P(s)$.

Theorem 4.9. *(Schur's Inequality) For non-negative x, y, z, n:*

$$x^n(x-y)(x-z) + y^n(y-z)(y-x) + z^n(z-x)(z-y) \geq 0$$

For $n = 1$ this expands to:

$$x^3 + y^3 + z^3 + 3xyz \geq x^2y + x^2z + y^2z + y^2x + z^2x + z^2y$$

This inequality is unique as it achieves equality when any of the following is true: $\{x = y = z\}$, $\{x = 0, y = z\}$, $\{y = 0, x = z\}$, $\{z = 0, x = y\}$.

The trick to solving inequality questions is often to use one of these inequalities and replace the variables with the ones you have. Here is an example.

Problem 4. If $a, b, c > 0$ prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$.

Proof. As $a, b, c > 0$, then $\frac{a}{b}, \frac{b}{c}, \frac{c}{a} > 0$ as they are just ratios of positive numbers. So, we can apply the AM-GM inequality on the three positive numbers $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$. What we get is:

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3} \geq \sqrt[3]{\frac{a}{b} \times \frac{b}{c} \times \frac{c}{a}}.$$

The right hand side cancels to give $\sqrt[3]{1} = 1$. So we get

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3} \geq 1.$$

Multiplying by three gives the desired result:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3.$$

□

Problems

4.4 Easy

1. [Squares] Farmer Brown wants to build a rectangular paddock, but he only has 1000m of fencing. What is the largest area he can enclose?
2. [Squares] Now Farmer Brown wants to build it next to a straight river, so that the fourth side can be the river. What is the largest area he can enclose now? Can you think of a geometric way of using the previous question to solve this one?
3. [Squares] If $x > 0$ prove that $x + \frac{1}{x} \geq 2$.
4. [AM-GM] Suppose $x > 0$. Show that
$$\frac{x^n}{1 + x + x^2 + \cdots + x^{2n}} \leq \frac{1}{2n+1}$$
5. [AM-GM] Let $n \in \mathbb{N}$. Prove that
$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} > n\left(\sqrt[n]{2} - 1\right)$$
6. [AM-GM] Prove that for all positive integers n,
$$\frac{(n+1)(2n+1)}{\sqrt[n]{(n!)^2}} > 6$$
7. [Rearrangement] Suppose $0 \leq a, b, c \leq 1$. Show that
$$2(a^3 + b^3 + c^3) - (a^2b + b^2c + c^2a) \leq 3$$
8. [Induction] Prove that for all natural numbers $n > 1$,
$$\sqrt{n+1} + \sqrt{n} - \sqrt{2} > 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}$$

4.5 Hard

1. [Rearrangement] Prove that for all $a, b, c \geq 0$,
$$\frac{(a+b+c)^2}{3} \geq a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}$$
2. [Induction] Show that for all integers $n > 2$,
$$\sqrt{n + \sqrt{(n-1) + \sqrt{(n-2) + \sqrt{\cdots + \sqrt{1}}}}} < \sqrt{n} + 1$$
3. [Substitution/C-S/Rearrangement] Let a, b, c be positive reals with $abc = 1$. Show that
$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1$$
4. [Cauchy-Schwarz] Suppose $p, q, r > 0$ and $pqr = 1$. Show that
$$\frac{1}{p^3(q+r)} + \frac{1}{q^3(r+p)} + \frac{1}{r^3(p+q)} \geq \frac{3}{2}$$

5. [AM-GM] Let x_1, x_2, x_3, x_4 be positive real numbers. Show that

$$\frac{x_1 + x_3}{x_1 + x_2} + \frac{x_2 + x_4}{x_2 + x_3} + \frac{x_3 + x_1}{x_3 + x_4} + \frac{x_4 + x_2}{x_4 + x_1} \geq 4$$

6. [Cauchy-Schwarz] Let a, b, c be positive reals. Prove

$$\frac{1}{a(a+b)} + \frac{1}{b(b+c)} + \frac{1}{c(c+a)} \geq \frac{27}{2(a+b+c)^2}$$