Algebra in Problem Solving (Junior)

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1 Algebra and Polynomials

You should have definitely by now seen algebra in class at school. Algebra is all about working with quantities that are yet unknown. For example, if you don't know a value, but you know that you will need to square it once you do, just consider x^2 where x is your unknown. This has the benefit that without knowing a quantity, you can still work on something and make whatever expressions are demanded by the problem. Once you know what x is you can then substitute and obtain the value of the expression.
Algebra is very useful. An algebraic expression can sometimes tell us something about an expression without even knowning the value of our variables. For example,

$$x^2$$

is always positive or zero, no matter what x is.
Another technique that can be used is graphing. Turning an algebraic problem into a graphical one. We will consider these techniques over the next few weeks.

Definition. A *polynomial* is a sum of non-negative powers of x. In general, a polynomial is written as

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

- n is the *degree* of the polynomial.
- $a_n, a_{n-1}, ..., a_1, a_0$ are the *coefficients* of the polynomial.
- a_n is the *leading coefficient*.
- The polynomial is *monic* if the leading coefficient is 1.
- A *root* to the polynomial is a solution to the equation $P(x) = 0$.

The coefficients can be negative, zero or even irrational if one likes. Examples of polynomials are

$$P(x) = x^2 - 3x - 2, Q(x) = -\frac{1}{2}x^5 - \pi x^4 + 2.34x^2 - x, R(x) = 4.$$

Note that polynomials do not contain terms involving fractional or negative powers of x, such as $\sqrt{x}$ or $\frac{1}{x}$.
Polynomials of degree 0, 1, 2, 3, 4, 5 are called constant, linear, quadratic, cubic, quartic and quintic, respectively.

1.1 Polynomial Operations

We can add and subtract polynomials by collecting like terms. We can multiply polynomials by multiplying each term of one with each term of the other. For example:

$$\begin{aligned}
(x^2 + 3x + 5) + (3x^2 - 2x - 7) &= (4x^2 + x - 2) \\
(x^3 + 3x - 2) - (x^3 - x^2) &= (x^2 + 3x - 2) \\
(x + 4) \times (x + 5) &= x^2 + 4x + 5x + 20 \\
&= x^2 + 9x + 20 \\
(x + 2) \times (x^2 - 4x + 3) &= x^3 - 4x^2 + 3x + 2x^2 - 8x + 6 \\
&= x^3 - 2x^2 - 5x + 6
\end{aligned}$$

But can we divide polynomials? It may not be obvious but we can. To divide $P(x)$ by $Q(x)$, we subtract multiples of $Q(x)$ for $P(x)$ until the remainder is small. We will define small to mean that the degree of the remainder must be less than the degree of $Q(x)$. Hence, if we're dividing by a linear polynomial, the

remainder will be constant, if we're dividing by a quadratic, the remainder will be at most linear. If the remainder is zero, then we say that $Q(x)$ *divides* $P(x)$, of $Q(x)$ is a *factor* of $P(x)$.
Some examples:

$$P(x) = 2x^2 + 3x + 7 \quad Q(x) = x + 1$$

We look for a multiple of $Q(x)$ close to $P(x)$. Because we're going to subtract it, and we want to make the remainder as simple as possible, we'll choose a multiple of $Q(x)$ that has the same leading term, $2x^2$.

$$(x + 1) \times (2x) = 2x^2 + 2x$$

The remainder is $P(x) - Q(x) \times (2x) = P(x) - (2x^2 + 2x) = x + 7$. This still has degree at least that of $Q(x)$, so we haven't finished. We find a multiple of $Q(x)$ which has the same leading term as that of $x + 7$, which will be just $(x + 1)$. Subtracting this from $(x + 7)$ gives 6. Hence, we look at what multiples of $Q(x)$ we subtracted, which were $2x$ and 1 times $Q(x)$, and thus:

$$P(x) = (2x + 1)Q(x) + 6$$

Another example, in slightly less detail:

$$P(x) = x^4 + 3x^3 - 4x^2 - 8x + 14 \quad Q(x) = x^2 - 3x + 2$$

Initially, we subtract $x^2Q(x)$, leaving $6x^3 - 6x^2 - 8x + 14$. We then subtract $6xQ(x)$, leaving $12x^2 - 20x + 14$. Finally, we subtract $12Q(x)$, leaving $16x - 10$. Thus,

$$P(x) = (x^2 + 6x + 12)Q(x) + (16x - 10)$$

Through this process, for any two polynomials $P(x)$ and $Q(x)$, we can write $P(x) = A(x)Q(x) + R(x)$, where $\deg(R(x)) < \deg(Q(x))$. If $R(x) = 0$, then we say that $Q(x)$ *divides* $P(x)$, or $Q(x)$ is a *factor* of $P(x)$.

1.2 Remainder Theorem

Let $P(x)$ be a polynomial and a is a constant. If we divide $P(x)$ by $(x - a)$ then the remainder has degree < 1 so is a constant. So we can write

$$P(x) = (x - a)Q(x) + r.$$

Substituting $x = a$ gives $P(a) = r$ and yields the remainder theorem.

Remainder Theorem

If $P(x)$ is divided by $(x - a)$ then the remainder is $P(a)$.

Factor Theorem

If $P(a) = 0$ then $(x - a)$ divides $P(x)$.

1.3 What kind of polynomials?

Let's look at the definition of a polynomial again.

$$P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

We've been pretty vague so far as to what the coefficients $a_n, a_{n-1}, \cdots, a_1, a_0$ actually are. If they all belong to some set S, we say that $P(x)$ is a polynomial over S. For example, if the coefficients are all integers, then we say that $P(x)$ is a polynomial over $\mathbb{Z}$, or that $P(x)$ is an *integer* polynomial. If the coefficients are all real, then $P(x)$ is a polynomial over $\mathbb{R}$, or that $P(x)$ is a *real* polynomial. (Other common sets are $\mathbb{Q}$ and $\mathbb{C}$, for rational and complex polynomials. Note also that any integer polynomial is a real polynomial, and likewise for other sets that are subsets of others).

We will consider real polynomials, unless specified otherwise. Note that the section on polynomial division above assumed that we can divide elements of the set by one another, and hence it is not valid for integer polynomials. (For an example, try dividing x^2 by $2x$. This won't work unless we consider them as rational or real polynomials).

1.4 Roots of Polynomials

A root of a polynomial $P(x)$ is a value a such that $P(a) = 0$. (By the factor theorem, this is the same as $(x - a)$ being a factor of $P(x)$.)
A double root of a polynomial $P(x)$ is a value a such that a is a root of $P(x)$, and a is a root of $\frac{P(x)}{(x-a)}$. This is equivalent to $(x - a)^2$ being a factor of $P(x)$.
We define the *multiplicity* of a root a to be the greatest integer m such that $(x - a)^m$ is a factor of $P(x)$, and $(x - a)^{m+1}$ is not a factor of $P(x)$. We call roots of multiplicity 1, 2, 3 to be single, double or triple roots.
The Fundamental Theorem of Algebra states that every non-constant polynomial over $\mathbb{C}$ has a root in $\mathbb{C}$. A corollary of this is that a polynomial over $\mathbb{C}$ of degree $n, n > 0$ has exactly n roots in $\mathbb{C}$ (with each root counted according to its multiplicity). For example, a polynomial of degree 3 could have three distinct roots, a double root and a distinct single root, or a triple root. This is equivalent to the statement that any polynomial $P(x)$ of degree n over $\mathbb{C}$ can be written as $P(x) = a(x - x_1)(x - x_2)\cdots(x - x_n))$ for some complex numbers $a, x_1, x_2, \cdots, x_n$.
Now, consider a real polynomial of degree $n, n > 0$. If it had more than n roots, then these would still all be roots when we considered the polynomial as a polynomial over $\mathbb{C}$. As it has exactly n roots when considered as a polynomial over $\mathbb{C}$, this would be a contradiction, and so a polynomial over $\mathbb{R}$ can have at most n roots in $\mathbb{R}$. (Likewise, polynomials over $\mathbb{Q}$ or $\mathbb{Z}$ can have at most n roots over these sets).

Another corollary is that if two polynomials are equal, then the corresponding coefficients are equal. ie, if

$$P(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_2x^2 + a_1x + a_0$$

$$Q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0$$

and $P(x) = Q(x)$, then $a_0 = b_0$, $a_1 = b_1, \cdots$, (and so $n = m$).

1.5 Quadratics

Quadratics are a particularly nice class of polynomials, because we can solve them fairly easily. Consider a polynomial $ax^2 + bx + c$, with roots α and β. We know that α and β have sum $-\frac{b}{a}$, and product $\frac{c}{a}$, so some quadratics, particularly monic ones, with 'nice' roots can be solved by inspection. For example, to find the roots of $x^2 + 7x + 10$, we need to find two numbers that add to -7 (**watch the signs!**) and multiply to 10. Checking the factors of 10, we find -2 and -5.
If the question is inconsiderate enough that no nice solutions present themselves, then we will need a method to solve the equation $ax^2 + bx + c = 0$. One way to solve this is to complete the square. Consider the following quadratic equation: $x^2 - 4x + 3 = 0$. To solve this by completing the square, we add a constant to both sides to make the left hand side a perfect square. Here, we need to add 1. Thus, $(x - 2)^2 = 1$. This means that $x - 2 = \pm 1$. Therefore x = 1 or x = 3.
For a general quadratic $ax^2 + bx + c$, the roots are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This is the quadratic formula, and is useful enough to justify remembering it. Similar formulae exist for cubics and quartics, but they are ridiculously complicated and *not* worth remembering. No such formula exists for polynomials of degree greater than 4.

1.6 Polynomial Identities

While these actually involve polynomials in two variables, they are common and useful enough to mention here.

$$
\begin{aligned}
(x+y)^2 &= x^2+2xy+y^2 \\
(x-y)^2 &= x^2-2xy+y^2 \\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3 \\
(x-y)^3 &= x^3-3x^2y+3xy^2-y^3 \\
x^2-y^2 &= (x+y)(x-y) \\
x^3+y^3 &= (x+y)(x^2-xy+y^2) \\
x^3-y^3 &= (x-y)(x^2+xy+y^2)
\end{aligned}
$$

There are similar identities to the latter ones for any exponent, as $(x-y)$ is a factor of x^n-y^n for any n, and $(x+y)$ is a factor of x^n+y^n for any odd n.

Problems

1.7 Easy

1. Which of the following expressions are polynomials?

$$4x^2 + x - 2,\ x^3 + 2x^2 - x - 1,\ \sqrt{x},\ 1,\ \frac{1}{x}.$$

2. Graph the following polynomials
   (a) $x^2 + 4x + 4$
   (b) $(x+1)(x-1)(x-3)$
3. The polynomial $a^2x^2 + (2a+1)x + 2$ has exactly one real root. Find a.
4. $\frac{x^2+3x+2}{2x+5} = k$. For which real k does this equation have at least one real solution for x?
5. What is $(x^2 - 4x + 3) \times (x+2)$?
6. Divide $x^5 + 3x^3 + 2x^2 + x + 1$ by $x^2 + x + 1$.
7. Using the fact that a polynomial of degree $n > 0$ over $\mathbb{C}$ has at most n roots, prove that if two polynomials are equal, then their corresponding coefficients are equal. (Hint: Subtract one polynomial from the other).
8. Find the roots of the following quadratics by completing the square:
   (a) $x^2 + 6x + 8$
   (b) $2x^2 + 5x + 2$
   (c) $5x^2 - 10 + 5$
9. Find all roots of the quartic $x^4 - 5x^2 + 4$
10. Use polynomial division to decide which of the following polynomials are multiples of $(x-2)$
    (a) $x^2 + 5x - 14$
    (b) $x^2 - x + 6$
    (c) $4x^2 - 5x + 6$
    (d) $3x^2 - 4x - 4$
    (e) $2x^3 + 3x^2 - 5x$
    (f) $x^3 - 4x^2 + 4x + 2$
11. Prove the identities in the last section.

1.8 Hard

1. What is the highest common factor of $x^4 + x^3 - x^2 + 1$ and $x^3 + 2x^2 + 2x + 1$.
2. By completing the square, prove the quadratic formula. ie, that the roots of the general quadratic $ax^2 + bx + c$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
3. *(Rational Root Theorem)* If the rational polynomial $P(x) = a_n x^n + a_{n-1} + ... + a_2 x^2 + a_1 x + a_0$ has root $\frac{p}{q}$ in simplest form, then p divides into a_0, and q divides into a_n.
4. Prove that if a real quadratic has non-real roots, then they exist in complex conjugate pairs. (ie, if $z = a + ib$ is a root of a real quadratic, then $\bar{z} = a - ib$ is also a root.)

5. Use polynomial division to decide which of the following polynomials are multiples of $x^2 - 3x + 2$

   (a) $x^3 + 2x^2 - 3x + 4$
   (b) $x^4 - 3x^3 + 3x^2 - 3x + 2$
   (c) $-x^4 + 2x^3 + 7x^2 - 20x + 12$
   (d) $3x^4 + 3x^3 + 2x^2 - 5x + 3$

6. Solve $x^3 - 7x^2 + 6 = 0$.

7. Prove that $a - b \mid P(a) - P(b)$ for any polynomial $P(x)$.

2 Functions

Last time, we learnt about polynomials, which involved lots of positive powers of x. But there is a lot more you can do with x. Polynomials as well as many other expressions involving x are called *functions*.
Often students are taught that functions are like vending machines. Essentially you put in a number, some cogs turn and the machine returns a (possibly different) number. Feel free to use this analogy. I prefer to think of a function as a pairing of numbers, where the first number in each pair is what goes into the function and the second number is what comes out. A more technical term for these pairs is a *relation* with one right number for each left number. We write $f(x)$ and say 'f of x' .

1. Is $f(x) = x^2$ a function?
2. Is $f(x) = \sqrt{x}$ a function?
3. Is $f(x) = \pm\sqrt{x}$ a function?

Many common functions include:

- Constant : $f(x) = c$
- Linear : $f(x) = ax + b$
- Quadratic : $f(x) = ax^2 + bx + c$
- Polynomial : $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$
- Square Root : $f(x) = \sqrt{x}$
- Exponential : $f(x) = e^x$
- Power : $f(x) = x^c$
- Logarithm : $f(x) = \ln(x)$
- Trigonometric : $f(x) = \sin(x)$ etc.
- Reciprocal : $f(x) = \frac{1}{x}$
- Floor : $f(x) = \lfloor x \rfloor =$ greatest integer less than or equal to x
- Fractional : $f(x) = \{x\} = x - \lfloor x \rfloor$
- Ceiling : $f(x) = \lceil x \rceil =$ smallest integer greater than or equal to x
- Absolute Value : $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

When discussing functions, it really is important to specify what kind of numbers can be put into a function. Sometimes you want to restrict what goes into the function for a particular reason. To be able to do this, let's have a quick recap of what sets are.

2.1 Sets

A set is a collection of elements without repetition. Here are a few examples of sets:

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ = the natural numbers or positive integers.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ = the integer numbers or just integers.
- $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}\}$ = the rational numbers. Also known as fractions (ratios of integers).

- $\mathbb{R}$ = the real numbers. All of the numbers you would find on a number line.
- $\mathbb{C}$ = the complex numbers.

There is a little bit of extra notation to understand about sets. For the explanations below assume $A = \{\text{cat}, \text{dog}\}$ and $B = \{\text{cat}\}$.

- When we write out the elements of a set, we put them inside curly braces, eg. $A = \{\text{cat}, \text{dog}\}$.
- $a \in A$ means a is an *element* of A. For example, cat $\in A$.
- $B \subset A$ means B is a *subset* of A. In our case, $B \subset A$ or $\{\text{cat}\} \subset A$.
- $A \cap B$ is the *intersection* of A and B, which is the common elements to both sets. In our case $A \cap B = \{\text{cat}\}$.
- $A \cup B$ is the *union* of A and B, which is all the elements in either set. In our case $A \cup B = \{\text{cat}, \text{dog}\}$.
- A^c is the *complement* of A, which is all the elements not in A. Here it is really important to know what universal set we are using. If for example our setting is the set of all pets, then A^c is the set of all pets that are not dogs or cats.

We will also sometimes add little symbols to well-known sets. For example,

- $\mathbb{R}^+$ is the set of all positive real numbers.
- $\mathbb{N}_0$ is the set of all natural numbers and zero.
- $\mathbb{Q}_0^-$ is the set of all non-positive rational numbers.

When working with the reals, we sometimes like to talk about intervals. Intervals are all the reals between two numbers. If we are talking about real numbers between a and b, we write it as:

- (a, b) = interval between a and b not including a and b.
- $(a, b]$ = interval between a and b not including a and including b.
- $[a, b)$ = interval between a and b including a and not including b.
- $[a, b]$ = interval between a and b including a and b.

2.2 Equations, expressions

Sometimes rather than being provided a function, you may instead have an expression. An *expression* is a mathematical statement. An equation is a mathematical expression that contains an equality operation (or as we call it an 'equals sign'). In algebra, very often you are going to be posed a problem where you want to solve an equation and more often than not, during your work you will encounter many different equations. Here are some tips to solving equations:

- Use techniques from high school mathematics: factorise, cancel, complete the square, use common identities.
- Consider it as a function and use functional techniques.
- Consider the graph of the equation and see if you can make any conclusions from that.
- Maybe there is no solution! Inequalities can be useful here.

2.3 Properties of Functions

Now that we are more familiar with functions, I would like to be a bit more technical. Functions are dependent on two very important sets: the *domain* and *codomain*. The domain is the set of all numbers (things) we can put into the function and the codomain is the set of all numbers (things) we are allowed to get out. If A is the domain and B is the codomain, we write $f : A \to B$.
The *range* is the set of all numbers that actually come out of the function. By definition, range $\subset$ codomain.
A function where every x value gives a unique y value is an *injective* or *one-to-one* function. In particular, $f(x) = f(y)$ then $x = y$. A function that is not injective is also called many-to-one. A function where you can get every value in the codomain is *surjective* or *onto*. In particular, there is a solution for the equation $f(x) = c$ for all $c \in$ codomain. A function that is both injective and surjective is called *bijective*.
We can *compose* functions, which is an interesting way of putting two functions together. Imagine taking x and putting it into f and then the result of that into g. That is called $g(f(x))$ or $(g \circ f)(x)$. When $g(f(x)) = f(g(x)) = x$, then we say that g is the *inverse* of $f : A \to B$ and write $g = f^{-1} : B \to A$.
There are other cool properties of certain functions:

- Even. $f(-x) = f(x)$ for all x.
- Odd. $f(-x) = -f(x)$ for all x.
- Periodic. $f(x + p) = f(x)$ for all x.
- Involution. $f(f(x)) = x$ for all x.
- Fixed point. These are points where $f(x) = x$.

Problems

2.4 Easy

1. Is $f : \mathbb{N} \to \mathbb{N}, f(x) = x^2$ the same function as $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$?
2. Is $f : \mathbb{R}^+ \to \mathbb{R}, f(x) = x^2$ the same function as $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$?
3. Is $f : \mathbb{R} \to \mathbb{R}_0^+, f(x) = x^2$ the same function as $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$?
4. What is the range of $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$?
5. What is the range of $f : \mathbb{N} \to \mathbb{N}, f(x) = x^2$?
6. What is the largest possible domain for the common functions given before?
7. What is a graphical test for injectivity?
8. For a surjective function, what is the relationship between codomain and range?
9. Find examples of injective and not injective functions.
10. Find examples of surjective and not surjective functions.
11. Find examples of bijective functions.
12. If $f(x) = 3x - 2$ and $g(x) = e^{7x}$, what is $f(g(x))$ and $g(f(x))$?
13. If $f(g(x)) = \frac{5}{\sqrt{3^{x+1}}}$, what are the possibilities for f and g?
14. What property must f have to have an inverse?
15. What properties do involutions have?
16. Do involutions have inverses and what are they?

2.5 Hard

1. Why is the inverse of a function the reflection of the function in $y = x$?
2. Find the inverses for each of the common functions.
3. Show that $p(x) = 2x^3 - 8x^2 + 3x - 2$ has at least one root. Give an interval of length at most 1 in which such a root lies.
4. Prove that there are two points on Earth which have the exact same temperature? [Hint consider points opposite each other]
5. Determine all increasing involutions.
6. Prove that you can always balance a rectangular table with four legs at a specific on uneven ground by simply rotating it.
7. Find all solutions $x \geq 1$ to the equation $x^3 = \lfloor x \rfloor^3 + \{x\}^3$.

3 Graphs

The thing that you draw with an x-axis and a y-axis is called the Cartesian plane. It is the 2D version of the real number line. It is supposed to be infinitely long in each direction, but usually we focus on a particular area of the Cartesian plane (and this is usually near the origin - the point where the axes meet). Points in this plane are of the form (x, y) where x and y are real numbers. Keeping this in mind, the graph of a function of a function $f(x)$ is $(x, f(x))$. We can write this in a more familiar form:

$$y = f(x).$$

Notice that for a function there is only one $f(x)$-value for every x, so on the graph there will only be one y-value for every x.

3.1 Shape of Graphs

Let's talk about the shape of functions for a little while. An *increasing* function f is such that $f(x) \geq f(y)$ whenever $x > y$. A *strictly increasing* function f is such that $f(x) > f(y)$ whenever $x > y$. Similarly, one can define *decreasing* and *strictly decreasing* functions. A function that is either increasing or decreasing is called *monotonic*.
Another very important concept is continuity. We say a function is *continuous* if we can draw its graph without taking the pen off the paper. More technically f is continuous at a if $\lim_{x \to a} f(x) = f(a)$. But we aren't going to worry about technicalities. Most of the functions you know are continuous. If a function is continuous we can do this awesome thing called:

Intermediate Value Theorem. *Suppose f is a continuous function defined on $[a_1, a_2]$ and $f(a_1) = b_1$ and $f(a_2) = b_2$. Then for any $b \in [b_1, b_2]$, there exists $x \in [a, b]$ such that $f(x) = b$.*

Think about what this means. It is saying that if the function is continuous and at some point it is equal to b_1 and somewhere it is equal to b_2, then somewhere in between it has to equal all of the numbers between b_1 and b_2. Amazing!
A function is *concave* if it looks like a sad face. If you draw a chord it should be completely below the graph. A *convex* function is like a happy face and the chord is completely above the graph. This can have implications for finding maxima and minima of functions.
The following are good techniques to consider when working with graphs:

- Combine graphs: add, subtract.
- Use properties of the common functions.
- Find roots (x-intercepts), y-intercepts, asymptotes.
- Figure out where it is positive, negative.
- Figure out where it is increasing, decreasing, concave, convex.
- Figure out where it is injective, surjective, bijective.
- Look out for other properties of functions: even, odd, periodic, involution, fixed point.
- Look for continuity and the Intermediate Value Theorem.

3.2 Quadratics

Quadratic functions are polynomials as we have already seen, so you should have a good understanding of them. The graph of a quadratic is called a parabola. It looks like the trajectory of a ball when you throw it. In fact, one cool property of it is that a parabola is the shape you get if you want to find all the points that are equally distant from a specific point and a specific line. When working with quadratics, it is important to be able to find the roots and the vertex. You can find the equation of the roots using the quadratic formula. If the parabola is

$$y = ax^2 + bx + c$$

then the vertex will be the point

$$\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right).$$

3.3 Interesting features of graphs

There are other cool properties of certain functions:

- Even. $f(-x) = f(x)$ for all x.
- Odd. $f(-x) = -f(x)$ for all x.
- Periodic. $f(x + p) = f(x)$ for all x.
- Involution. $f(f(x)) = x$ for all x.
- Fixed point. These are points where $f(x) = x$.

These features dictate what happens to the graph of the function $f(x)$. It is left to the reader to find a graphical interpretation of these features.

Problems

3.4 Easy

1. Think of a graphical way of determining whether or not a graph represents a function.
2. Graph all the types of functions listed in the functions handout.
3. Consider how the graph transforms if we go from $y = f(x)$ to

   - $y = af(x)$,
   - $y = f(x + c)$,
   - $y = f(bx)$,
   - $y = f(x) + d$, or
   - $y = af(bx + c) + d$.

   Try this for a few of the types of functions from the previous question. Focus on what happens graphically.

4. With the help of a graph

   (a) Define a strictly decreasing function.
   (b) Prove that a strictly monotonic function is injective.
   (c) Is a monotonic function surjective?
   (d) Give an example of a non-continuous function.

5. What are the three cases that arise from considering the discriminant?
6. Can you ever get three solutions from a quadratic equation?

3.5 Hard

1. What property must f have to have an inverse?
2. What properties do involutions have?
3. Do involutions have inverses and what are they?
4. The section about interesting features of graphs left it up to you the reader, to work out the graphical implication of each feature. Do this.

4 Inequalities

At school, we mostly work with equations. These are two expressions separated by the equality symbol, =, called 'equals'. However, in mathematics it often pays to talk about inequations. If we want to say two things are unequal we can stick an inequality sign, $\neq$, between them. This is kind of cool, but not quite strong enough. We would like to be able to say which expression is bigger or smaller. To do this we use the following symbols:

- $<$ is less than
- $>$ is more than
- $\leq$ is less than or equal to
- $\geq$ is more than or equal to

All inequalities will involve these four symbol $>$, $<$, $\geq$ or $\leq$. Here they are in action:

$$5 > 3, \quad 2 < 7, \quad 8 \geq 1, \quad 4 \leq 3 + 1.$$

Note, that it is ok to say $8 \geq 1$ because even though 8 is more than 1 it is also more than or equal to 1, by definition.

4.1 Solving Inequalities

As you can probably guess we will not be working just with numbers but with algebra and inequalities. So, just like with equalities, we need to be able to solve inequalities. We know how to solve $2x + 1 = 5$, but how do we solve $2x + 1 > 5$?
Well, here are the things you can do with inequalities:

- Add or subtract things from both sides of an inequality. So, if I start with $2x + 1 > 5$, I can happily get: $2x > 4$ (subtracting 1 from both sides), or $-4 > -2x$ (subtracting $2x + 5$ from both sides).
- Multiply or divide both sides of an inequality. CAUTION: if you multiply or divide by a negative value, the inequality sign flips (from $<$ to $>$ and vice versa or from $\leq$ to $\geq$ or vice versa).
- You have to be careful with other algebraic manipulations. In general, you can't square both sides. The reason is that on one side you might be squaring a negative number (which should flip the inequality), but on the other side you might be squaring a positive number (which does not flip the inequality)! The rule of thumb is that if both sides are the same sign (positive or negative) you can square. Just remember that if both sides were negative do not forget to flip the inequality.
- Other algebraic manipulations such as exponentiation or logarithms you can do, but many others like sine or cosine you cannot. You will learn in time how to recognise which ones you can and can't use.

I now provide a few examples of simple solving of inequalities:

Problem 1. Solve $2x + 1 < 5$.

Proof. Subtracting 1 from both sides yields $2x < 4$. Dividing both sides and remembering that we do not flip the inequality because 2 is positive, gives

$$x < 2.$$

This is the answer. All numbers x which are less than 2 satisfy the inequality. □

Problem 2. Solve $\frac{x+1}{x-1} \leq 2$.

Proof. Note, the inequality is $\leq$ not $<$, so do not forget to keep using the correct symbol. The first thing we would like to do is multiply by $x-1$, but we need to be careful. What if it is negative? So, we have two cases:
Case 1: $x-1<0$. This is the same as saying $x<1$. Now, when we multiply both sides by $x-1$ we have to flip the inequality so we get

$$x+1 \geq 2x-2.$$

Combining a few steps into one, I now subtract x from both sides and add 2 to both sides giving me

$$3 \geq x \text{ or equivalently } x \leq 3.$$

So, in this case where $x<1$ we get that $x \leq 3$, which together implies all x that are less than 1 ($x<1$).
Case 2: $x-1 \geq 0$. This is the same as saying $x \geq 1$. Now when we multiply both sides by $x-1$ we leave the inequality sign so we get

$$x+1 \leq 2x-2.$$

Again applying standard algebraic techniques we get

$$3 \leq x \text{ or equivalently } x \geq 3.$$

So, in this case where $x \geq 1$, we get $x \geq 3$, so together this implies $x \geq 3$.
Thus, the answers for this inequality are $x<1$ and $x \geq 3$. □

4.2 The Strongest Inequality

From here on in, we will no longer concentrate on solving inequalities. In much the same way that in school maths, you have finished learning how to solve maths equations, after a little bit of practice, inequalities should be easy to solve too.
What we do now is prove that certain inequality statements are always true for all values of our variables. A lot of this is based on one very simple inequality that we all know and that is especially strong:

Theorem 4.1.

$$x^2 \geq 0 \textit{ for all } x \in \mathbb{R}.$$

This says that all real numbers squared are nonnegative (positive or zero). Pretty straightforward.
Let's have a look at how powerful this is.

Problem 3. Prove that for all real numbers a and b,

$$a^2+b^2 \geq 2ab.$$

Proof. This should be shocking. Why should the left hand side always be bigger than the right hand side? Well here is why. We know from our strongest inequality that any real number squared is nonnegative. Therefore,

$$(a-b)^2 \geq 0.$$

But this expands to $a^2-2ab+b^2 \geq 0$, which when we rearrange becomes

$$a^2+b^2 \geq 2ab.$$

□

4.3 Useful Inequalities

There are many very useful inequalities that have been proved for us using complicated techniques.

Theorem 4.2. *(Discrete Inequality) If $a, b \in \mathbb{Z}$ and $a > b$, then $a \geq b + 1$.*

This is easy to understand. If your variables are whole numbers and one is bigger than the other, then it must be at least one bigger than the other!

Theorem 4.3. *(Rearrangement Inequality) Let $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$ be real numbers and let $z_1, z_2, \ldots, z_n$ be any permutation of $y_1, y_2, \ldots, y_n$. Then*

$$x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1 \leq x_1 z_1 + x_2 z_2 + \cdots x_n z_n \leq x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Again, this one can be understood in a common sense way. It is easiest to understand with an example scenario. Someone is offering you money. You are shown three piles of money which contain \$10 notes, \$20 notes and \$50 notes respectively. You are told that you are allowed to take 1 note from one pile, 2 notes from another pile and 5 notes from another pile. Obviously, the worst (smallest) choice is to take 1 \$50 note, 2 \$20 notes and 5 \$10 notes, while the best option is to take 5 \$50 notes, 2 \$20 notes and 1 \$10 note.
What we've got are two sequences: \$10, \$20, \$50 and 1, 2 and 5. If we pair them up in order we get the biggest return. If we pair them up in reverse order we get the worst result. Whereas any other pairing is in between. This is the rearrangement inequality.

Theorem 4.4. *(AM-GM-HM Inequality) For positive real numbers $x_1, x_2, \ldots, x_n$,*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

with equality iff $x_1 = x_2 = \cdots = x_n$.

Theorem 4.5. *(Cauchy-Schwarz Inequality) For* all *real numbers $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$,*

$$(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2 \leq (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2)$$

Equality holds iff $y_1 = r x_1$, $y_2 = r x_2, \ldots, y_n = r x_n$ for some real constant r.
There is an alternative version that is perhaps easier to use, but it works only for positive *reals $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$:*

$$(\sqrt{x_1 y_1} + \sqrt{x_2 y_2} + \cdots + \sqrt{x_n y_n})^2 \leq (x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n)$$

The trick to solving inequality questions is often to use one of these inequalities and replace the variables with the ones you have. Here is an example.

Problem 4. If $a, b, c > 0$ prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$.

Proof. As $a, b, c > 0$, then $\frac{a}{b}, \frac{b}{c}, \frac{c}{a} > 0$ as they are just ratios of positive numbers. So, we can apply the AM-GM inequality on the three positive numbers $\frac{a}{b}, \frac{b}{c}, \frac{c}{a}$. What we get is:

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3} \geq \sqrt[3]{\frac{a}{b} \times \frac{b}{c} \times \frac{c}{a}}.$$

The right hand side cancels to give $\sqrt[3]{1} = 1$. So we get

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3} \geq 1.$$

Multiplying by three gives the desired result:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3.$$

□

Problems

4.4 Easy

1. Solve the following two questions:
   (a) $\frac{x+2}{x} > 3$
   (b) $\frac{x+2}{x} \leq 3$

   What is the relationship between the answers to (a) and (b)?
2. [Squares] Farmer Brown wants to build a rectangular paddock, but he only has 1000m of fencing. What is the largest area he can enclose?
3. [Squares] Now Farmer Brown wants to build it next to a straight river, so that the fourth side can be the river. What is the largest area he can enclose now?
4. [Squares] Prove that $a^2 + b^2 \geq 2ab$. When does equality hold?
5. [Squares] If $a, b, c > 0$ and $a + b + c = 2$, prove that $ab + bc \leq 1$.
6. [Squares] If $x > 0$ prove that $x + \frac{1}{x} \geq 2$.
7. [AM-GM-HM] If $a, b > 0$ prove that $(a + b)(\frac{1}{a} + \frac{1}{b}) \geq 4$.
8. [AM-GM-HM] If $a, b, c > 0$ prove that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$.
9. [AM-GM-HM] If $a, b, c > 0$ and $abc = 1$, show that $(a + b)(b + c)(c + a) \geq 8$.
10. [Rearrangement] Prove that
$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

4.5 Hard

1. [AM-GM-HM] Let a, b, c be positive numbers. Prove that
$$(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$$
2. [AM-GM-HM] Think of a generalisation for the above problem. Prove it.
3. [AM-GM-HM] Let x and y be positive real numbers such that $x + y = 1$. Show that
$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \geq 9$$
4. [Rearrangement] If a, b, c are positive reals, prove that
$$a^4b + b^4c + c^4a \geq a^3bc + b^3ac + c^3ab$$
5. [Rearrangement] If a, b, c are positive reals, show that
$$(ab)^2 + (bc)^2 + (ca)^2 \geq abc(a + b + c)$$
6. [Cauchy-Schwarz] If a, b, c, d are positive reals, with $a + b + c + d = 64$, prove that
$$\frac{1}{a} + \frac{1}{b} + \frac{4}{c} + \frac{16}{d} \geq 1$$

7. [Cauchy-Schwarz] If a, b, c, d are positive reals, then show

$$\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} + \sqrt{4d+1} < 6$$

when $a + b + c + d = 1$.

8. [Cauchy-Schwarz] If x, y, z are reals with $8x - 9y + 12z = 10$, find the minimum value of $x^2 + y^2 + z^2$. For what values of x, y, z does this occur?