THE INSTABILITY OF A TOROIDAL
MAGNETIC GEON AGAINST
chapter 7   GRAVITATIONAL COLLAPSE

KIP S. THORNE

I. THE PROBLEM

This conference has called renewed attention to the problem of very massive, spherically symmetric bodies which, according to Einstein's theory of relativity, collapse past the gravitational radius and on into a singularity. In this paper we shall consider a similar situation: the gravitational collapse of bodies made not of matter but of electromagnetic fields—geons.

The motivation for studying collapsing geons is most certainly *not* that they might play a role in quasi-stellar radio sources; far from it. There is not the slightest reason to believe that geons exist in nature or could be constructed by man. Rather, our motivation is that the analysis of a collapsing geon is much simpler than the analysis of a collapsing sphere of matter in one important respect: The relation between pressure and energy density for the electromagnetic field is more precisely known and much easier to work with than the equation of state for nuclear matter at high densities—a mixture of heavy nuclei, electrons, neutrons, and even hyperons. Consequently, a geon permits the study of the nature of gravitational collapse as a phenomenon within Einstein's theory of relativity without entangling one in the uncertainties and complexities of the equation of state.

The Einstein-Maxwell equations for interacting electromagnetic and gravitational fields allow the existence of many types of geons (Wheeler 1955, 1962). Wheeler has emphasized that all geons are unstable against gravitational collapse, electromagnetic explosion, or leakage of electromagnetic radiation out of the active region, and that those which undergo gravitational collapse should be very useful in studying the issue of the final state of collapsed systems.[1] However, only recently has the study of geon collapse been pursued in earnest.

The initial studies of geon collapse (M. A. Melvin and J. A. Wheeler, unpublished) involved a toroidal magnetic geon (a toroidal bundle of magnetic-field lines bound together against the disruptive Maxwell-Faraday pressures by their mutual gravitational attraction). It soon became clear that the analysis would be greatly simplified by con-

KIP S. THORNE, Palmer Physical Laboratory, Princeton University.

[1] See his 1963 Les Houches lectures (Wheeler 1964) for a detailed exposition of this theme.

sidering the limiting case in which the major axis of the torus is infinite. Thus, Melvin and Wheeler were led to consider a cylindrically symmetric bundle of magnetic-field lines of infinite length. By using Weyl's theory of axially symmetric gravitational fields (Weyl 1919; Weyl and Bach 1922; Levi-Civita 1919), Melvin succeeded in constructing a cylindrical *static* geon of this type, in which the Maxwell-Faraday pressures are *precisely* counterbalanced by the gravitational forces (Melvin 1964a). This solution was at first believed to be *unstable* against *spontaneous* gravitational collapse or electromagnetic explosion.[2] However, subsequent analyses have argued for stability! (Melvin 1964b; Thorne 1964). No perturbation of the system, which vanishes outside a finite region around the symmetry axis, can lead to gravitational collapse or electromagnetic explosion. This surprising result appears to stem from the fact that Melvin's system is *not* a highly concentrated accumulation of electromagnetic energy as it appears at first glance; rather, it is the most *diffuse* distribution of electromagnetic and gravitational energy possible under the circumstances. Melvin's system is *not* a *geon;* it is a magnetic *universe.*

Melvin's magnetic universe presents many interesting problems, but since they are apparently unrelated to gravitational collapse, this is not the place to discuss them. Nevertheless, we can learn a lesson from the problems which the early misinterpretation of Melvin's solution caused: Although the *mathematics* of toroidal geons is simplified by letting the major radius become infinite and thereby obtaining cylindrical geons, the *interpretation* of the solutions so obtained becomes vastly more difficult. The Newtonian approximation, the most powerful guide we have to the interpretation of solutions of Einstein's equations, is applied easily only in an *asymptotically flat* space. For example, in order to define the mass of an object in an unambiguous manner, one must get sufficiently far away that orbits about it obey Kepler's laws to a close approximation. This occurs only in an approximately flat region of space. But one can never get so far away from a cylindrical geon that space becomes asymptotically flat. Just as an infinite line charge creates a logarithmically diverging term in the electrostatic potential of Maxwell theory, so a source of infinite length creates diverging terms in the metric of Einstein's theory. The logarithmic term causes no special problems in electrostatics because of the linearity of the theory. But in the non-linear Einstein theory the divergent terms become real impediments to the interpretation of solutions.

For these reasons it is appropriate to turn away from the study of cylindrical geons and return to a toroidal geon of the type originally considered by Wheeler and Melvin. The analysis of the full dynamical behavior of a toroidal geon would be quite difficult. Fortunately, however, it is not necessary to perform the *full* analysis in order to verify the instability of the geon and get some insight into its collapse. For the purposes of this discussion we need only examine the geon at a moment of time symmetry.

What we propose to do is to examine, at a moment of time symmetry, a sequence of toroidal magnetic geons all having the same total flux Φ and the same *proper* major circumference $2\pi b'$, but having different *proper* minor radii a'. We will determine the mass M of the geon as measured by an external observer, and the geometry of the spacelike hypersurface of time symmetry, as functions of Φ, b', and a'. From these features of the system we will infer that toroidal magnetic geons with sufficiently small minor radii are unstable against gravitational collapse. Finally, we will compare these toroidal geons with a collapsing cloud of dust and from the similarity between the two at the moment of time symmetry, we will argue that their dynamical evolution should be similar.

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[2] The belief in its instability was strongly supported by numerical calculations of the dynamics of the system when perturbed, which were performed by the author and reported at the Symposium on Gravitational Collapse. Because these calculations have since proved irrelevant to the issue of gravitational collapse, they are reported elsewhere (Thorne 1964) rather than here.

II. THE SOLUTION

The analysis of a system at a moment of time symmetry requires the solution of the initial value equations of general relativity

$$R_\mu{}^0 - (\tfrac{1}{2})\delta_\mu{}^0 R = 8\pi G T_\mu{}^0$$

analogous to div $\boldsymbol{E} = 4\pi\rho$ and div $\boldsymbol{B} = 0$ in electromagnetism (Darmois 1927; Stellmacher 1937; Lichnerowicz 1955; Foures-Bruhat 1956). At a moment of time symmetry these four equations reduce to one:

$$^{(3)}R = 16\pi G T_0{}^0 \,,$$

where $^{(3)}R$ is the scalar curvature of the hypersurface of time symmetry (Brill 1959). Brill has shown that when one has axial symmetry as well as time symmetry, one can put the metric on the hypersurface of time symmetry in the form

$$d\sigma^2 = \psi^4[e^{2q}(d\rho^2 + dz^2) + \rho^2 d\phi^2]$$

and that the initial value equation then becomes

$$4(\nabla^2\psi)/\psi + \nabla^2 q - (1/\rho)(\partial q/\partial\rho) = -8\pi G\psi^4 e^{2q}T_0{}^0 \,.$$

Here ∇^2 is the *flat-space* Laplacian operator.

Brill has found that the quantity q ("gravitational wave factor") is often associated with gravitational radiation (Brill 1959; Wheeler 1964). Because we wish to minimize the amount of gravitational radiation present, and because we want our initial value equation to be as simple as possible, we set $q = 0$.[3] In this case the metric is

$$d\sigma^2 = \psi^4 d\sigma_1{}^2 = \psi^4(d\rho^2 + dz^2 + \rho^2 d\phi^2) \,, \tag{1}$$

and the initial-value equation reads

$$\nabla^2\psi = -2\pi G\psi^5 T_0{}^0 \,. \tag{2}$$

Here $d\sigma_1{}^2$ is a Euclidean "base metric," and $d\sigma^2$, the actual metric of the problem, is conformally flat. The initial value equation is just Poisson's equation with $-2\pi G\psi^5 T_0{}^0$ as the source of the "conformal correction factor" ψ.

We now further specialize the time-symmetric and axially symmetric geometry to the case of a toroidal magnetic geon (Fig. 1). Let the magnetic field lines be entirely contained within a torus of major radius b and minor radius a, as measured in the base metric and let $b \gg a$ ("slender ring"). Let the magnetic field lines be so distributed in the torus that $2\pi G\psi^5 T_0{}^0$ is uniform throughout it.[4] Finally, let the coordinate r measure base-metric distance from the guiding center of the torus

$$r = [(\rho - b)^2 + z^2]^{1/2} \,.$$

[3] Setting $q = 0$ does not actually remove *all* gravitational radiation. If it did, outside the source of the gravitational field the solution to the initial-value equation would also be a static solution to the full-vacuum field equations.

[4] This condition is not necessary to the analysis. The solution in the case of a non-uniform source leads to a conformal correction factor ψ which is identical to that of eq. (3) outside the geon, if that equation is rewritten in terms of the base-metric quantities ρ, z, and b. However, *inside* the geon the more general case gives a more complicated expression for ψ. Since the added complications yield no new insights, we avoid them.

Under these conditions the form of the solution to the initial value equation for ψ (eq. [2]) can be seen directly from the electrostatics of a charged circular ring: Far away the electric potential varies as (total charge)/(distance to ring), while nearby it varies as $2 \times$ (line density of charge) $\times$ log (distance to center of wire). Similarly, the behavior of the conformal correction factor far from the geon is governed by (observed mass of geon)/(distance to geon), while near the edge of the torus it is governed by (magnetic mass-energy per unit length) $\times$ log (distance to guiding line of torus).[5] Note that the analogy is *not* a complete one. For the geon the scale of lengths changes as one moves from the near region to the far region. Consequently, the *observed mass*, which governs the gravitational pull far away, is *not* equal to the line integral of the *magnetic mass-energy per unit length*, which governs the gravitational pull nearby. What a difference from elementary electrostatics!
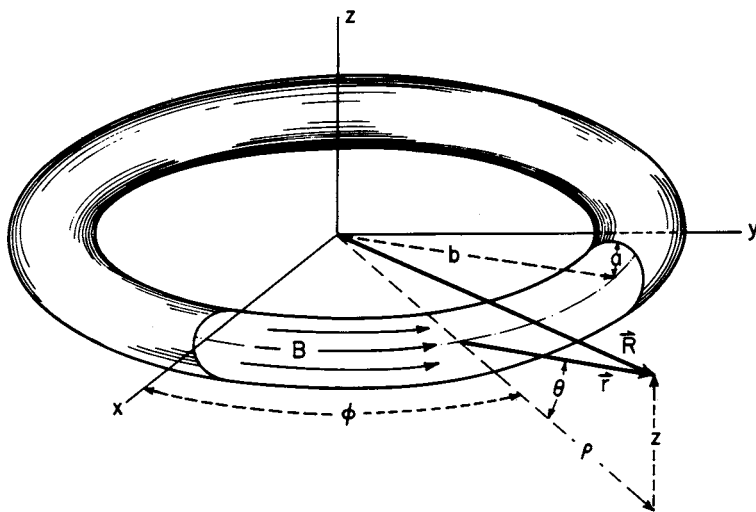

FIG. 1.—A toroidal magnetic geon at a moment of time symmetry as seen in the base metric. The magnetic-field lines thread the interior of the torus and are confined entirely within it. The electric field vanishes at the moment of time symmetry, but the time rate of change of the electric field ordinarily does not ($\dot{E} = \text{curl } B$).

We turn now from a qualitative description of the solution to a precise statement of it. Set $c = G = 1$. The solution external to the geon can be put in the form

$$\psi(R) = 1 + \frac{1}{\pi} \frac{M_m}{M} \frac{M_m}{[z'^2 + (\rho' + b')^2]^{1/2}} \text{K} \left(\left[\frac{4\rho' b'}{z'^2 + (\rho' + b')^2} \right]^{1/2} \right)$$

$$\approx 1 + M/(2R), \quad \text{when} \quad R \equiv (\rho^2 + z^2)^{1/2} \gg b ,$$

$$\approx 1 + \frac{M_m}{M} \frac{M_m}{2\pi b'} \log \left(\frac{8b'}{r'} \right), \quad \text{when} \quad a' < r' \ll b' ,$$

(3)

where K is the complete elliptic integral of the first kind. The masses appearing here are M, the mass measured by an external observer, and M_m, the magnetic mass obtained by integrating $(B^2/8\pi)$ over the physical volume of the torus. The terms ρ', r', and z' are renormalized values of ρ, r, and z constructed so as to make the metric flat inside the torus:

$$\rho' = \psi_o^2 \rho, \qquad r' = \psi_o^2 r, \qquad z' = \psi_o^2 z,$$

where $\psi_o = \psi(r = 0)$. We have yet to write down the conformal correction factor ψ for the interior of the geon. In the case where $b' \gg a'$ ("slender ring") and $\psi^5 T_0{}^0 =$ (constant throughout torus), which we are considering, both the conformal correction factor ψ and the physical magnetic field B (as measured in a locally Lorentz reference frame) turn out to be essentially uniform throughout the torus.

$$\psi(\mathbf{R}) = \psi_o = M_m/M. \qquad\qquad \text{(\textbf{R} inside torus)} \quad (4)$$

The masses which appear in our equations can be expressed in terms of the total magnetic flux $\Phi = B\pi a'^2$, the proper minor radius $a' = a\psi_o^2$, and the proper circumference $2\pi b' = 2\pi b\psi_o^2$. The externally observed mass, M, is

$$M = M_m + M_p \qquad\qquad (5)$$

where M_m, the "magnetic mass," and M_p, the "gravitational potential mass," turn out to be

$$M_m = (\Phi^2 b')/(4\pi a'^2)$$
$$= (B^2/8\pi) \times \text{(proper volume of torus)} \qquad\qquad (6)$$

and

$$M_p = -M_m \frac{M_m}{2\pi b'} \log\left(\frac{8b'}{a'}\right).$$

It is interesting to note that, if we had calculated the mass of the geon in the Newtonian approximation (Wheeler 1964), we would have obtained precisely the same result, providing we had remembered that the magnetic pressure, $B^2/8\pi$, and energy density, $B^2/8\pi$, *both* act as sources for and feel the gravitational field. M_m is just the "mass" obtained by integrating the energy density over the torus; and M_p is just the gravitational potential energy of a Newtonian geon.

Let a particular toroidal geon (characterized by Φ, a', and b') be observed as it evolves away from its moment of time symmetry. Will it explode, or will it collapse? To get some insight into this question, consider a sequence of geons all having the same total flux Φ and major circumference $2\pi b'$, but having different minor radii, a'. Plot their masses as functions of a' (Fig. 2). For large a' where the magnetic mass dominates, the measured mass varies as $1/a'^2$. However, as a' decreases, the negative gravitational potential mass becomes more and more important, and it eventually wins out causing the measured mass to decrease as $-1/a'^4$ (second term in eq. [5] dominates over first term).

Turn now from a family of geons to the behavior of a particular geon as it evolves away from its configuration of time symmetry. If a' is sufficiently large ("dilute field configuration"), we can use ordinary Maxwell theory to solve the problem, and we will find, of course, that the geon flies apart because of the magnetic pressure. On the other hand, if a' is sufficiently small ($<10^{-5}$ light years in the case of Fig. 2), the gravitational attraction between the magnetic field lines should dominate over the electromagnetic repulsion, and the geon should collapse. In either case, as the geon evolves away from its configuration of time symmetry, the changing magnetic field creates electric fields.

A precise description of the gravitational collapse of a toroidal geon must await the full solution of the dynamical equations. However, we can make a few general statements about collapse dynamics at this point. As has been discussed by Wheeler (1964), there are two possible modes of collapse: (1) a mode in which the magnetic field lines collapse onto the guiding center of the torus (minor radius shrinks to zero as major radius remains finite) and (2) a mode in which the field lines all contract to a point (major and minor radii simultaneously shrink to zero). Wheeler shows that mode (2) is a very reasonable mode of gravitational collapse. On the other hand, two facts suggest
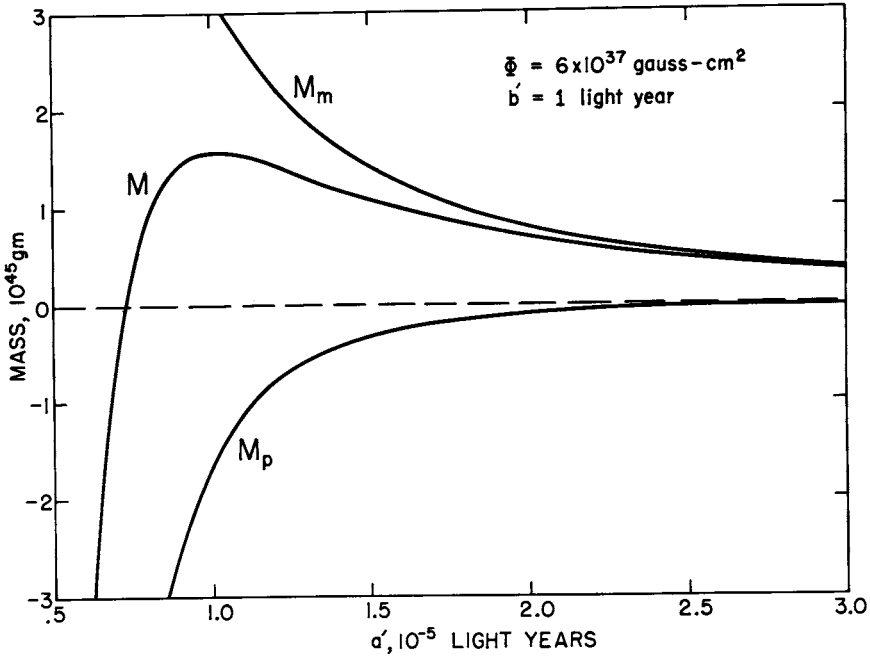

FIG. 2.—The magnetic mass, gravitational potential mass, and measured mass of a time-symmetric toroidal magnetic geon as functions of the configuration at the moment of time symmetry. Here we consider geons all having the same flux $\Phi = 6 \times 10^{37}$ gauss-cm², and the same proper circumference $2\pi b' = 2\pi$ light-years, but having different proper minor radii a'. Since $M_m =$ (constant depending on b' and Φ) $\times (b'/a')^2$ and $M_p = -$(another constant depending on b' and Φ) $(b'/a')^4 \log (8b'/a')$, these curves will have the same shape independently of Φ and b'. In particular, as a' decreases the observed mass, M, will always rise slowly, reach a maximum, then fall off rapidly toward $-\infty$.

that mode (1) might never occur: First, there are two cases on record[6] in which magnetic flux is capable of *preventing* gravitational collapse; but (to the author's knowledge) there are no known examples of the collapse of a magnetic-flux-containing-and-conserving system. Second, we shall see later that in the limit $M = 0$, Φ, b', a' finite, the time-symmetric hypersurface of a toroidal magnetic geon is curved up into a closed universe with 3-sphere topology. If the geon were to collapse onto its guiding line, one might expect the topology of a 3-torus instead.

Because the maximum in the curve for mass versus minor radius should mark the point separating exploding geons from collapsing geons, it is of interest to examine the

[6] The two cases are Melvin's magnetic universe (Melvin 1964*a;* Thorne 1964) and the Reissner-Nordstrom solution for a "wormhole" threaded by an electromagnetic field (Wheeler 1964).

"critical geons" corresponding to this maximum. We do so in Table 1. Here we have an indication of how preposterous it would be to believe geons exist in nature or could be created and studied experimentally. To be near criticality a laboratory-size geon must have field strengths corresponding to supranuclear energy densities, while a geon of more reasonable field strengths must be fantastically large.

The huge magnetic field encountered in the laboratory-size geon reminds us that the equation of state for the electromagnetic field becomes very complicated when a field strength of the order of the characteristic field of electron pair theory is reached:

$$F_{\text{crit}} = mc^2/[e(\hbar/mc)] = 4.4 \times 10^{13} \text{ gauss} = 1.3 \times 10^{18} \text{ volt/m}$$

(Euler and Kockel 1935; Heisenberg 1936). When such field strengths are reached, vacuum polarization, virtual pair production, and occasionally even real pair production occur. Then we are back in the realm of elementary particle physics and all the complica-

TABLE 1

VALUES OF PROPER CIRCUMFERENCE, PROPER MINOR RADIUS, MAGNETIC FLUX, MAGNETIC FIELD STRENGTH, MAGNETIC ENERGY DENSITY, AND MASS FOR SEVERAL "CRITICAL GEONS"*

	Laboratory-size Geon	Astrophysical-size Geon	Universe-size Geon
Proper circumference, $2\pi b'$	60 m	6 light-years	6×10^{10} light-years
Proper minor radius, a'	10^{-2} cm	10^{-5} light-years	10^5 light-years
Magnetic flux, Φ	6×10^{22} gauss cm^2	6×10^{37} gauss cm^2	6×10^{47} gauss cm^2
Magnetic field strength, B	2×10^{26} gauss	2×10^{11} gauss	20 gauss
Magnetic energy density, $B^2/8\pi$	1.6×10^{30} gm/cm^3	1.6 gm/cm^3	1.6×10^{-20} gm/cm^3
Magnetic mass, $M_m = (B^2/8\pi) \times$ volume	3×10^{30} gm	3×10^{45} gm	3×10^{55} gm
Gravitational potential mass, M_p	-1.5×10^{30} gm	-1.5×10^{45} gm	1.5×10^{55} gm
Measured mass, M	1.5×10^{30} gm	1.5×10^{45} gm	1.5×10^{55} gm

* Geons which are expected to collapse if squeezed but explode if distended; or, more precisely, geons corresponding to a maximum in the M versus a' curve.

tions which it adds to the collapse phenomenon, to escape from which we turn to the study of geons. Fortunately, however, we are free to consider large geons as well as small ones, so that if, in following the collapse of a particular geon, one finds uncomfortably large field strengths developing in the center, he need only turn to a much larger geon in which a given stage of collapse is characterized by much smaller field strengths.

Return to the curve of observed mass versus minor radius for fixed flux and major radius (Fig. 2). There is one disturbing thing about this curve: the mass M goes *negative* for minor radius $a' < 0.72 \times 10^{-5}$ light-years. *This is impossible*, according to a theorem of Brill (1959) on the positive-definiteness of mass in time-symmetric, axially symmetric systems. What is wrong here? The answer is that our solution (eqs. [3]–[6]) is *not physically admissible* in those cases where the mass M is negative. Whenever M is negative the conformal correction factor ψ is *not* everywhere positive; it contains a nodal 2-surface surrounding the geon. As was pointed out by Brill, such a nodal surface in the space of the base metric is a single point of the hypersurface of time symmetry, since $d\sigma^2 = 0$ everywhere on it. The region outside the node of ψ is totally cut off from the geon, and the geometries of both the exterior and interior regions exhibit cusplike singularities at the node. Configurations of negative mass are thus ruled out.

It is still puzzling that as the minor radius a' is decreased toward a certain finite value (0.72×10^{-5} light-years in the case of Fig. 2), the observed mass M approaches arbi-

tarily close to zero. To understand this phenomenon better, we examine the geometry of the hypersurface of time symmetry as a' approaches a'_o, that value for which the observed mass M vanishes. Figure 3 presents "imbedding diagrams" of the geometry of the ρ-ϕ 2-surface for geons with successively decreasing values of a'. It is seen from this figure that, as a' approaches a'_o, the throat between the exterior region and the geon closes off leaving, in the limit of $a' = a'\zeta$, (1) a geon so dense that it has closed space up around itself, and (2) a perfectly flat, empty exterior region.

I

$a' = 3.0 \times 10^{-5}$ l.y.
$M = 0.34 \times 10^{45}$ gm

II

$a' = 10^{-5}$ l.y.
$M = 1.55 \times 10^{45}$ gm

III

$a' = 0.75 \times 10^{-5}$ l.y.
$M = 0.41 \times 10^{45}$ gm
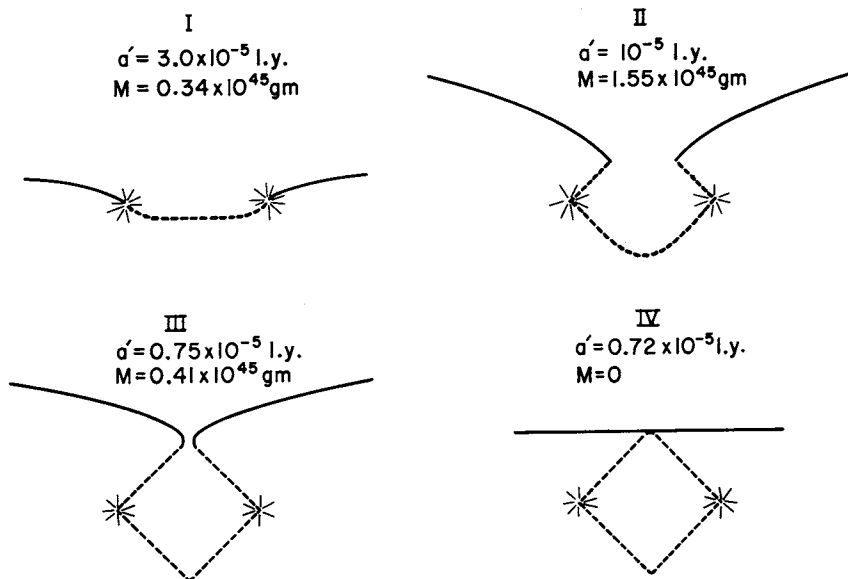
IV

$a' = 0.72 \times 10^{-5}$ l.y.
$M = 0$

FIG. 3.—"Imbedding diagrams" of the geometry of the $\rho - \varphi$ 2-*surface* for geons all having the same flux $\Phi = 6 \times 10^{37}$ gauss-cm² and proper circumference $2\pi b' = 2\pi$ light-years, but having different proper minor radii a'. In these diagrams we plot ($\frac{1}{2}\pi$) (proper circumference of a circle about the geon's center) $= \rho\psi^2(\rho) \equiv \Re$ horizontally against a fictitious vertical coordinate ζ. ζ is so constructed that in the region where the curve is solid the proper length of a radial line segment is $d\sigma = (d\Re^2 + d\zeta^2)^{1/2}$ but in the region where the curve is dotted $d\sigma = (d\Re^2 - d\zeta^2)^{1/2}$. (Thus, two points separated by a 45° *dotted* line are actually very close together.) In more physical terms, if one rotates the figures about the vertical axis one obtains from the solid parts of the curves the $\rho - \varphi$ 2-surface as it looks imbedded in a Euclidean 3-space, and from the dotted parts the $\rho - \varphi$ 2-surface as it looks imbedded in a Minkowski 3-space with "imaginary" vertical axis. The regions in which the magnetic field is located are indicated by radiating lines.

From the diagrams it is clear that as a' decreases toward $a'_o = 0.72 \times 10^{-5}$ light-years (the value for which the observed mass vanishes), the geon pinches off from the exterior region, leaving, in the limit $a' = a'_o$, a Lorentz-flat exterior (Diagram IV). The geon which is separated from the exterior space at $a' = a'_o$ is so dense that it curves space up around itself into a closed universe. At the point in this closed universe farthest from the geon's center the hypersurface of time symmetry is flat.

Although this figure specializes to the case $\Phi = 6 \times 10^{37}$ gauss-cm² and $b' = 1$ light-year, the imbedding diagrams would be essentially unchanged if Φ and b' were chosen differently.

This situation is strikingly similar to the one depicted in Figure 4 for a collapsing cloud of dust (Oppenheimer and Snyder 1939; Klein 1962; Beckedorff 1962; Beckedorff and Misner 1964). The topologies of the hypersurfaces of time symmetry are identical in the two cases; and in both cases the same type of pinch-off occurs as the surface area of the energy-containing region is reduced.[7] The similarity between the hypersurfaces of

[7] It is striking that in the pinch-off the entire torus, including its central "hole," separates from the external universe. One might have expected the pinch-off to occur in a canal-like region around the ring. A similar situation arises when one has two Schwarzschild solutions very close together: They share a

time symmetry in the two cases is so striking that one is led to expect that the dynamical behavior of a collapsing geon will resemble that of a collapsing dust cloud. In particular, it seems likely that, as the geon collapses, an event horizon similar to the "Schwarzschild singularity" will develop.

The detailed investigation of the dynamics of the collapse of a toroidal magnetic geon —free of all reference to any equation of state—should give still further insight into gravitational collapse in general and the issue of the final state.

I

S = 308km²

M= 0.84 M☉

(1/10 Scale)

II

S = 20km²

M=0.43M☉

(1/2 Scale)

III

S = 1.35km²

M=0.027M☉
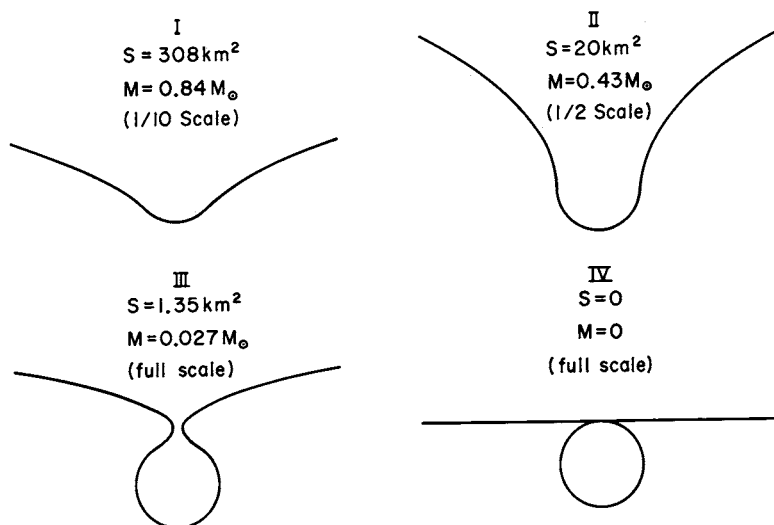
(full scale)

IV

S = 0

M = 0

(full scale)

Fig. 4.—"Imbedding diagrams" of the geometry of the hypersurface of time symmetry for collapsing clouds of dust. For each dust cloud, if one takes a two-dimensional surface of constant azimuthal angle φ at the moment of time symmetry and imbeds it in Euclidean 3-space, one obtains the surface formed by rotating the corresponding curve about its vertical axis. Four diagrams are shown corresponding to clouds of dust all containing 1 $M\odot$ of matter ($\int T_0{}^0 dv_{\text{physical}} = M\odot$) but having different surface areas. As the surface area shrinks from ∞ to 0, the mass of the cloud as measured from the exterior shrinks from 1 $M\odot$ to 0. In the limit where the surface area reaches zero, the dust cloud pinches off from the exterior region and one is left with a flat, empty exterior universe and a separate closed Friedman universe viewed at the moment of time symmetry.

The dust cloud is located in the circular region of each diagram, while the parabolic region corresponds to empty space outside the cloud's surface.

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common throat which opens out into the universe, and it is only *below* the common throat that they separate from each other (Brill and Lindquist 1963; cf. especially their Fig. 3).

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