THE ASTROPHYSICAL JOURNAL, Vol. 159, March 1970

NONRADIAL PULSATION OF GENERAL-RELATIVISTIC STELLAR MODELS. V. ANALYTIC ANALYSIS FOR $l = 1$*

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Received 1969 August 9

ABSTRACT

The theory of small, adiabatic, dipole perturbations of a star away from hydrostatic equilibrium is developed within the framework of general relativity. The analysis is linearized in the perturbation amplitudes. The "odd-parity" perturbations describe differential rotation, while the "even-parity" perturbations describe pulsation. In the pulsational case there are two degrees of freedom associated with the fluid motion and *no* degrees of freedom in the gravitational field (no dipole gravitational waves). During the pulsation the star's external gravitational field is completely unperturbed, to first order in the amplitude, despite the fact that its surface performs a finite, back-and-forth motion.

In both the pulsational and rotational cases, the Einstein field equations are put into simple forms suitable for numerical integration.

I. INTRODUCTION AND SUMMARY

The first four papers in this series (Thorne and Campolattaro 1967; Price and Thorne 1969; Thorne 1969*a*, *b*; hereinafter called Papers I, II, III, and IV, respectively) treated the quadrupole and higher-multipole perturbations ($l \geq 2$) of relativistic stellar models. The dipole perturbations ($l = 1$), which are treated in this paper, require a different type of analysis from the others, because (i) there can be no dipole gravitational waves and (ii) the tensor spherical harmonics for $l = 1$—unlike those for $l \geq 2$—satisfy the algebraic identities

$$\chi^1{}_{Mjk} = 0 \,, \quad \Phi^1{}_{Mjk} + \Psi^1{}_{Mjk} = 0 \tag{1}$$

(cf. Appendix A of Paper I; also Table 1 of this paper).

To physicists unfamiliar with the theory of Newtonian stellar pulsations, it might seem surprising that dipole pulsations are possible. Doesn't the equality of gravitational and inertial mass rule out such pulsations altogether? No; it *does* rule out any oscillatory motion of the star's center of mass, and it *does* rule out any emission of dipole gravitational waves, but it *does not* rule out dipole pulsations. Imagine, for example, a small nuclear explosion at a point off the center of the star. The matter at the star's center will clearly be driven into motion away from the explosive center. If the explosion does not disrupt it, the star will subsequently pulsate with its center of mass fixed relative to the distant stars (conservation of momentum), but with the matter at its geometric center in motion. If this stellar pulsation is resolved into normal modes, it *must* contain modes with $l = 1$ because for $l = 1$ the central matter moves, whereas for $l = 0$ and $l \geq 2$ the central matter cannot move ($\delta r = 0$ at $r = 0$).

In this paper we first treat (§ II) $l = 1$ perturbations with "odd parity," i.e., $\pi = (-1)^{l+1} = +1$. As in the case of $l \geq 2$, these perturbations describe stellar rotation

* Supported in part by the National Science Foundation [GP-15911 (formerly GP-9433), GP-9114], the Office of Naval Research [Nonr-220(47)], and the U.S. Air Force Office of Scientific Research [AFOSR 68-1453A].

rather than stellar pulsation. There are no pulsations, because to pulsate the star's fluid must experience an oscillatory perturbation in its pressure; such a perturbation must have scalar-spherical-harmonic angular dependence; and there are no scalar spherical harmonics with odd parity.

Most of this paper (§ III) is devoted to $l = 1$ perturbations with "even parity," i.e., $\pi = (-1)^l = -1$. A specific gauge is introduced for the analysis of these perturbations (§ III*a*). This gauge, which differs from that of Regge and Wheeler (1957), is defined only up to arbitrary, time-dependent displacements of the origin of coordinates (§ III*b*).

In this gauge the fluid motions are described by two amplitudes—$W(t, r)$ for radial displacements, and $V(t, r)$ for azimuthal displacements—and the gravitational perturbations are described by three metric functions—$H_0(t, r)$, $H_1(t, r)$, $H_2(t, r)$. The Einstein field equations, which govern the pulsations, consist of initial-value equations for the gravitational perturbations, H_A, in terms of the fluid motions, W and V; plus dynamical equations for the time evolution of W and V. Consequently, the star has only two dynamical degrees of freedom, and both are associated with the fluid motion (§ III*c*).

With an appropriate choice of gauge—i.e., that choice which puts the origin of coordinates at the star's center of mass, but which permits finite fluid motions at $r = 0$—the external gravitational field has $H_0 = H_1 = H_2 = 0$. Consequently, the geometry of spacetime outside the star is unaffected by the stellar pulsations; it remains the spherically symmetric, Schwarzschild geometry of the unperturbed star (§ III*d*).

The Einstein field equations plus necessary boundary conditions form a well-posed mathematical framework for calculating the stellar pulsations. Numerical integrations on a computer should not be much more difficult than in the Newtonian case (§ III*g*).

In this paper, as in Papers I–IV, most of the mathematical derivations are confined to appendices. The body of the paper concentrates on a precise, self-contained presentation of the main results of the analysis.

The notation is that introduced in Paper I [including a choice of units in which $c = G = 1$, and including the use of Greek indices to run from 0 to 3 (i.e., over x^0, x^1, x^2, x^3) and Latin indices to run from 2 to 3 (i.e., over $x^2 = \theta$ and $x^3 = \phi$)]. Equations from Papers I–IV are denoted thus: equation (I, 19), equation (II, 7), etc. All perturbations are treated to first order in the amplitude. The letter M is used to represent both the total mass-energy of the unperturbed star and the "projection index" of spherical harmonics. This should not be confusing, since group theory guarantees that the spherical-harmonic index M will *never* appear in the initial-value or dynamical equations that govern the perturbations. Only the star's mass M can appear there.

II. ODD-PARITY PERTURBATIONS[1]

With an appropriate choice of gauge (cf. Appendix A) the spacetime geometry for a star, undergoing a general $l = 1$, $\pi = (-1)^{l+1} = +1$ perturbation, takes the stationary form

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) - 2r^2\omega dt[\Sigma_j(4\pi/3)^{1/2}\Phi^1{}_{Mj}dx^j] \,. \quad (2)$$

Here ν and λ are functions of r, which describe the unperturbed geometry; ω is a function of r, which describes the perturbation; and $\phi^1{}_{Mj}$ is a vector spherical harmonic (cf. Table 1). This geometry is necessarily stationary (i.e., the metric is independent of the time coordinate t), by virtue of the Einstein field equations.

The fluid in the perturbed star has four-velocity

$$u^0 = e^{-\nu/2} \,, \quad u^r = 0 \,, \quad u^j = -\Omega e^{-\nu/2}(4\pi/3)^{1/2}\Phi^1{}_M{}^j \,. \quad (3)$$

Here Ω is a function of r only, which describes the perturbation. The density and pressure as functions of radius r are unperturbed. Physically, the fluid perturbation is differential

[1] For mathematical details, see Appendix A.

rotation without any change (to first order in angular velocity) in the star's shape, density, or pressure.

When $M = 0$, the fluid has (cf. Table 1 and eq. [3])

$$u^\phi/u^0 = d\phi/dt = \Omega(r) \,, \quad u^\theta/u^0 = 0 \,, \tag{4}$$

which corresponds to differential rotation about the polar axis (z-axis) with an angular velocity Ω that is a function of r only. The metric for $M = 0$ has

$$-g_{0\phi}/g_{\phi\phi} = \omega \,, \quad g_{0\theta} = 0 \,; \tag{5}$$

consequently, ω is the "angular velocity of the local inertial frames" as discussed by Hartle (1967).

When $M = \pm 1$, the real part of the perturbation corresponds to differential rotation about the $\mp \boldsymbol{e}_x$ axis with angular velocity $\Omega/\sqrt{2}$; and the imaginary part corresponds to differential rotation about the $-\boldsymbol{e}_y$ axis with the same angular velocity, $\Omega/\sqrt{2}$. (We adopt the usual convention that $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, and $z = r \cos\theta$.) Also, for $M = \pm 1$, the angular velocity of the local inertial frames has a real part of magnitude $\omega/\sqrt{2}$ and direction $\mp \mathbf{1}_x$, and an imaginary part of magnitude $\omega/\sqrt{2}$ and direction $-\mathbf{1}_y$.

TABLE 1

TENSORIAL SPHERICAL HARMONICS FOR $l = 1$*

Type	Parity	Components	
Scalar....	Even	$Y^1{}_0 = (3/4\pi)^{1/2} \cos\theta$	$Y^1{}_{\pm 1} = \mp(3/8\pi)^{1/2} e^{\pm i\phi} \sin\theta$
Vector...	Even	$\Psi^1{}_{0\theta} = -(3/4\pi)^{1/2} \sin\theta$	$\Psi^1{}_{\pm 1\theta} = \mp(3/8\pi)^{1/2} e^{\pm i\phi} \cos\theta$
		$\Psi^1{}_{0\phi} = 0$	$\Psi^1{}_{\pm 1\phi} = -i(3/8\pi)^{1/2} e^{\pm i\phi} \sin\theta$
	Odd	$\Phi^1{}_{0\theta} = 0$	$\Phi^1{}_{\pm 1\theta} = i(3/8\pi)^{1/2} e^{\pm i\phi}$
		$\Phi^1{}_{0\phi} = -(3/4\pi)^{1/2} \sin^2\theta$	$\Phi^1{}_{\pm 1\phi} = \mp(3/8\pi)^{1/2} e^{\pm i\phi} \sin\theta \cos\theta$
Tensor...	Even	$\Phi^1{}_{M\theta\theta} = -\Psi^1{}_{M\theta\theta} = Y^1{}_M$	
		$\Phi^1{}_{M\phi\phi} = -\Psi^1{}_{M\phi\phi} = \sin^2\theta \, Y^1{}_M$	
		$\Phi_M{}^1{}_{\theta\phi} = -\Psi^1{}_{M\theta\phi} = 0$	
	Odd	$\chi^1{}_{Mjk} = 0$ for all j, k	

* These formulae are calculated from the definitions given in Appendix A of Paper I. The tensor indices of these harmonics are raised and lowered with the two-sphere metric γ_{jk}: $\gamma_{\theta\theta} = 1$, $\gamma_{\phi\phi} = \sin^2\theta$, $\gamma_{\theta\phi} = 0$.

To construct a stellar model with $l = 1$ differential rotation, one first constructs a nonrotating equilibrium model (cf. Thorne 1967). One then specifies the fluid angular velocity $\Omega(r)$ in an arbitrary manner, subject only to the constraint that

$$\Omega \ll (m/r^3)^{1/2} \,, \tag{6}$$

where $m(r) = \frac{1}{2} r(1 - e^{-\lambda})$ is the mass inside radius r. (This constraint guarantees that the centrifugal force will not deform the star so badly that structural changes of order $\Omega^2 r^3/m$ must be included in the analysis.) One finally calculates the dragging of inertial frames (i.e., the function $\omega(r)$ which Ω generates) by integrating the differential equation

$$r^{-4}[r^4 e^{-(\lambda+\nu)/2} \omega_{,r}]_{,r} + 4r^{-1}[e^{-(\lambda+\nu)/2}]_{,r}(\omega - \Omega) = 0 \tag{7}$$

subject to the boundary conditions (cf. Hartle 1967)

$$\begin{aligned} \omega &= \text{finite constant} + O(r^2) \qquad &\text{at} \quad r = 0 \,, \\ \omega &= 0 &\text{at} \quad r = \infty \,. \end{aligned} \tag{8}$$

For rigid rotation about the polar axis (Ω independent of r, and $M = 0$), the above equations and integration procedure have been discussed in great detail by Hartle

(1967); and numerical integrations have been carried out by Hartle and Thorne (1968). The aspects of their analysis which are not discussed above can easily be extended to our case of differential rotation with Ω a function of r but not of θ.

III. EVEN-PARITY PERTURBATIONS

a) The Metric and Fluid Perturbations[2]

With an appropriate choice of gauge (cf. Appendix B) the spacetime geometry for a star undergoing $l = 1$, $\pi = (-1)^l = -1$ perturbations takes the form

$$ds^2 = e^\nu(1 + H_0 Y^1{}_M)dt^2 + 2H_1 Y^1{}_M dtdr - e^\lambda(1 - H_2 Y^1{}_M)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (9)$$

The metric perturbation functions H_0, H_1, H_2 are functions of r and t, so that in this case —by contrast with the odd-parity case—the gravitational field is dynamical rather than static. This metric differs from that for $l \geq 2$ (eq. [I, 7b]) in two respects: (i) for $l \geq 2$ $H_0 = H_2$, but here $H_0 \neq H_2$; (ii) for $l \geq 2$ there are perturbations of $g_{\theta\theta}$ and $g_{\phi\phi}$, but here there are not.

The fluid element originally at (r, θ, ϕ) in the unperturbed star is displaced to $(r + \xi^r, \theta + \xi^\theta, \phi + \xi^\phi)$ in the perturbed star, where

$$\xi^r = r^{-2}e^{-\lambda/2}W(r, t)Y^1{}_M, \quad \xi^j = -r^{-2}V(r, t)\Psi^1{}_M{}^j. \quad (10)$$

Hence, the function $W(r, t)$ describes the radial motion of the fluid, while $V(r, t)$ describes its tangential motion.

In discussing the fluid perturbations, we shall often make use of the Lagrangian change in pressure, Δp, which results from the fluid displacement (10) and metric perturbation (9).

$$\Delta p = -e^{-\nu/2}S(r, t)Y^1{}_M,$$
$$S \equiv \gamma p e^{\nu/2}(r^{-2}e^{-\lambda/2}W_{,r} + 2r^{-2}V - \tfrac{1}{2}H_2) \quad (11)$$

(cf. eqs. [I, 16] and [I, C3]). Here $\gamma(r)$ is the adiabatic index and $p(r)$ is the pressure of the unperturbed configuration.

b) Gauge Arbitrariness[3]

The gauge used in Paper I for describing even-parity, $l \geq 2$ pulsations is uniquely determined. However, the analogous gauge (eq. [9]) for $l = 1$ is not. The forms (9) and (10) of the metric and fluid perturbations are left unaffected by infinitesimal coordinate transformations of the form

$$x^{\alpha'} = x^\alpha + \eta^\alpha, \quad (12a)$$

where

$$\eta_0 = -a_{,t}fY^1{}_M, \quad \eta_1 = ar^{-1}e^\lambda fY^1{}_M, \quad \eta_j = af\Psi^1{}_{Mj}, \quad (12b)$$

$$f = r\exp\left[-\int_r^\infty r^{-1}(1 - e^\lambda)dr\right], \quad a = a(t) = \text{arbitrary function of time}. \quad (12c)$$

Such a change of gauge produces the following changes in the perturbation functions:

$$H_0' = H_0 + (2a_{,tt}e^{-\nu} + ar^{-1}\nu_{,r})f, \quad (13a)$$

$$H_1' = H_1 + a_{,t}[2r^{-1}(1 - e^\lambda) - \nu_{,r}]f, \quad (13b)$$

$$H_2' = H_2 - ar^{-1}[2r^{-1}(1 - e^\lambda) + \lambda_{,r}]f, \quad (13c)$$

$$W' = W - are^{\lambda/2}f, \quad (13d)$$

[2] Cf. Appendix B. [3] Cf. Appendix C.

$$V' = V + af\,, \tag{13e}$$

$$S' = S\,. \tag{13f}$$

c) The Field Equations[4]

The Einstein field equations for the metric and fluid perturbations (9)–(11) reveal the following:

There are only two dynamical degrees of freedom in the star. These are associated with the radial fluid displacement ($W[t, r]$) and the azimuthal fluid displacement ($V[t, r]$). The hyperbolic differential equations which determine the time evolution of W and V are

$$\begin{aligned} &r^{-2}(\rho + p)e^{(\lambda-\nu)/2}W_{,tt} + \tfrac{1}{2}(\rho + p)e^{\nu/2}(r^{-2}e^{-\lambda/2}\nu_{,r})_{,r}W \\ &- 2r^{-2}(\rho + p)(e^{\nu/2})_{,r}V - S_{,r} \\ &- (\rho + p)e^{\nu/2}[\tfrac{1}{2}H_{0,r} + (\tfrac{1}{2}\nu_{,r} - r^{-1})H_0 + (\tfrac{1}{4}\nu_{,r} + r^{-1})H_2] = 0\,, \end{aligned} \tag{14a}$$

$$(\rho + p)e^{-\nu}V_{,tt} + e^{-\nu/2}S + r^{-2}e^{-\lambda/2}p_{,r}W - \tfrac{1}{2}(\rho + p)H_0 = 0\,. \tag{14b}$$

Notice the similarity to two of the three dynamical equations (eqs. [I,9b,c]) for $l \geq 2$.

The perturbation functions S, H_0, and H_2, which appear in these dynamical equations, are fixed in terms of W and V by the initial-value equations

$$\begin{aligned} &H_{2,r} + [r^{-1}(1 + e^{\lambda}) - \lambda_{,r} + 4\pi r e^{\lambda}(\rho + p)]H_2 \\ &\quad = 8\pi r e^{\lambda}[(\rho + p)r^{-2}e^{-\lambda/2}W_{,r} + r^{-2}e^{-\lambda/2}\rho_{,r}W + 2r^{-2}(\rho + p)V]\,, \end{aligned} \tag{15a}$$

$$S = \gamma p e^{\nu/2}(r^{-2}e^{-\lambda/2}W_{,r} + 2r^{-2}V - \tfrac{1}{2}H_2)\,, \tag{15b}$$

$$\begin{aligned} &H_{0,r} + [\nu_{,r} - r^{-1}(2 - e^{\lambda})]H_0 \\ &\quad = -r^{-1}H_2 + 8\pi r e^{\lambda}(Se^{-\nu/2} + r^{-2}e^{-\lambda/2}p_{,r}W)\,; \end{aligned} \tag{15c}$$

and the additional metric perturbation function H_1 is determined by

$$H_1 = -rH_{2,t} + 8\pi(\rho + p)e^{\lambda/2}W_{,t} \tag{15d}$$

or, equivalently (modulo the other field equations), by

$$H_{1,t} = e^{\nu}H_{0,r} + (\tfrac{1}{2}\nu_{,r} - r^{-1})e^{\nu}H_0 + (\tfrac{1}{2}\nu_{,r} + r^{-1})e^{\nu}H_2\,. \tag{15d$'$}$$

d) The External Gravitational Field

Equations (15a), (15c), and (15d) are easily solved in the vacuum outside the pulsating star, where[5]

$$\nu = -\lambda = \ln(1 - 2M/r)\,. \tag{16}$$

The general solution is

$$H_0 = \tfrac{1}{3}\frac{\zeta^3\beta + 8M^2\beta_{,tt}}{\zeta(1 - \zeta)^2}\,, \quad H_1 = -\frac{2M\zeta}{(1 - \zeta)^2}\beta_{,t}\,, \quad H_2 = \frac{\zeta^2}{(1 - \zeta)^2}\beta\,, \tag{17a}$$

where

$$\zeta \equiv 2M/r\,, \quad \beta = \beta(t) = \text{arbitrary function of } t\,.$$

This general vacuum perturbation can be set to zero by the choice of gauge of equations (12) with $\alpha(t) = -\tfrac{2}{3}M\beta(t)$. Consequently, *the geometry of spacetime outside the pulsating star is completely unperturbed; it remains the spherically symmetric Schwarzschild geometry!*

[4] Cf. Appendix D.

[5] Here, and everywhere except when it is an index on a spherical harmonic, M represents the total mass of the star.

e) Removal of Gauge Arbitrariness

The result of the last section provides an attractive method for removing the arbitrariness of gauge discussed in § III*b*: *Of all coordinate systems in which the metric takes the form of equation (9), there will be one and only one (aside from time translations and rotations) for which* $H_0 = H_1 = H_2 = 0$ *outside the star. We shall henceforth restrict ourselves to this particular coordinate system.* All other coordinate systems of the type (9) can be obtained from it by a change of gauge of the form (12), which will yield the perturbed external metric (17a) with

$$\beta(t) = (3/2M)\alpha(t) . \tag{17b}$$

The changes of gauge (12) have a simple geometric interpretation: They correspond to displacing the coordinate system by the amount

$$\delta x^{\alpha} = -\eta^{\alpha}(t, r, \theta, \phi)$$

relative to its original position. Near $r = \infty$ this coordinate displacement has components

$$\delta t = \alpha_{,t} r Y^1{}_M , \quad \delta r = \alpha Y^1{}_M , \quad \delta x^j = \alpha r^{-1} \Psi^1{}_M{}^j ; \tag{18}$$

and in terms of Cartesian coordinates with

$$z = r \cos\theta , \quad x + iy = r \sin\theta e^{i\phi}$$

its components are

$$\delta t = \left(\frac{3}{4\pi}\right)^{1/2} \alpha_{,t} z , \quad \delta x = \delta y = 0 , \quad \delta z = \left(\frac{3}{4\pi}\right)^{1/2} \alpha \quad \text{for} \quad M = 0 , \tag{19a}$$

$$\delta t = \left(\frac{3}{8\pi}\right)^{1/2} \alpha_{,t}(\mp x - iy) , \quad \delta x = \mp i \delta y = \mp \left(\frac{3}{8\pi}\right)^{1/2} \alpha , \quad \delta z = 0 \quad \text{for} \quad M = \pm 1 . \tag{19b}$$

This displacement is simply a translation of the coordinate system through a distance $(3/4\pi)^{1/2}\alpha(t)$ relative to its original position. The non-Galilean δt associated with this time-dependent translation of the space coordinates has the form one would expect from the Lorentz transformations of special relativity.

f) Boundary Conditions on the Field Equations

The perturbation functions H_0, H_1, H_2, W, V, and S must satisfy certain boundary conditions at the center and surface of the star. At the center the perturbations must correspond to a finite displacement of the fluid, $\xi^r \sim$ const. $\times\ Y^1{}_M$ and $\xi^j \sim r^{-1}\Psi^1{}_M{}^j$, with a well-behaved Lagrangian change in pressure, $\Delta p = -e^{-\nu/2} S\, Y^1{}_M \sim r Y^1{}_M$. By combining these constraints with the field equations (14) and (15), one finds that the perturbation functions have the following forms at the star's center:

$$\begin{aligned} W &= w r^2 + O(r^4) , \quad V = -w r + O(r^3) , \\ H_0 &= h r + O(r^3) , \quad H_1 = 8\pi(\rho_c + p_c) w_{,t} r^2 + O(r^4) , \\ H_2 &= \frac{8\pi}{5}\left\{\frac{(\rho_c + p_c)^2}{\gamma_c p_c}\left[e^{-\nu_c} w_{,tt} + 4\pi\left(\frac{\rho_c}{3} + p_c\right) w + \frac{h}{2}\right] \right. \\ &\quad \left. + 2w\left[\frac{d\rho}{d(r^2)}\right]_c\right\} r^3 + O(r^5) , \\ S &= (\rho_c + p_c) e^{\nu_c/2}[e^{-\nu_c} w_{,tt} + 4\pi(\tfrac{1}{3}\rho_c + p_c) w + \tfrac{1}{2}h] r + O(r^3) . \end{aligned} \tag{20}$$

Here w and h are functions of time, and the subscript c refers to the value of a quantity at the center of the unperturbed star.

Notice that equations (20) are qualitatively the same as in Newtonian theory: there are two arbitrary functions of time in the asymptotic expansion about the star's center, corresponding to the two degrees of freedom in the fluid motion. The function $w(t)$ determines the displacement of the fluid element that originated at the star's center; i.e., it determines $W(r = 0, t)$ and $V(r = 0, t)$, and thence $\xi(r = 0, t)$. The functions $h(t)$ and $w(t)$ together determine the amount by which fluid elements near the center are compressed; i.e., they determine $S(r \approx 0, t)$, and thence the Lagrangian changes in density and pressure.

At the star's surface the Lagrangian change in pressure, $\Delta p = -Se^{-\nu/2}Y^1{}_M$, must vanish; consequently,

$$S = 0 \quad \text{at} \quad r = R_- \,, \quad \text{i.e., just inside the star's surface .} \tag{21a}$$

In addition, the geometry of spacetime inside the star must join smoothly to the geometry of spacetime outside. More particularly, the first and second fundamental forms of the star's three-dimensional timelike surface must be the same when measured from the star's exterior, where $H_0 \equiv H_1 \equiv H_2 \equiv 0$, as when measured from the star's interior. Straightforward but nontrivial calculations show that the first fundamental form is continuous if and only if

$$H_0 = 0 \quad \text{at} \quad r = R_- \,, \tag{21b}$$

and the second fundamental form is continuous if and only if

$$H_1 = 0 \qquad \text{at} \quad r = R_- \,, \tag{21c}$$

$$H_2 = 8\pi r^{-1}e^{\lambda/2}W\rho \qquad \text{at} \quad r = R_- \,, \tag{21d}$$

$$H_{0,r} = -8\pi r^{-2}e^{\lambda/2}(1 + \tfrac{1}{2}r\nu_{,r})W\rho \qquad \text{at} \quad r = R_- \,. \tag{21e}$$

The surface boundary conditions (21) are not all independent of the field equations (14) and (15). In fact, *the field equations together with two boundary conditions (eq. [21b] on H_0 and eq. [21e] on $H_{0,r}$) guarantee that all of the other boundary conditions are satisfied.* Since the field equations are a fourth-order differential system in r, and since there are two independent boundary conditions at the star's surface, precisely two arbitrary functions of time are needed to fix the solution near the surface. We can take them to be the amplitudes, $W(r = R, t)$ and $V(r = R, t)$, for the radial and tangential components of the fluid displacement.

Notice that the qualitative features of the surface boundary conditions (two constraints on gravitational field; two degrees of freedom in fluid displacement) are precisely the same in general relativity theory as in Newtonian theory.

g) Normal Modes of Pulsation

Numerical analyses of the normal modes of dipole pulsation for relativistic stellar models should not be much more complicated than numerical analyses of the normal radial modes. The normal modes will have perturbation functions of the form

$$\begin{aligned} H_0 &= H_0(r)e^{i\omega t}\,, \quad & H_1 &= H_1(r)e^{i\omega t}\,, \quad & H_2 &= H_2(r)e^{i\omega t}\,, \\ W &= W(r)e^{i\omega t}\,, \quad & V &= V(r)e^{i\omega t}\,, \quad & S &= S(r)e^{i\omega t}\,. \end{aligned} \tag{22}$$

Equations (14a), (14b), (15a), (15b), and (15c) will become coupled eigenequations for the eigenfunctions $H_0(r)$, $H_2(r)$, $W(r)$, $V(r)$, and $S(r)$. This system of eigenfunctions will be of fourth order. For any given value of the frequency, ω, two of the four independent solutions, corresponding to the two independent constants $we^{-i\omega t}$ and $he^{-i\omega t}$ in equation (20), will be physically acceptable at the star's center, and two solutions will be unacceptable. At the star's surface, two solutions will be physically acceptable, while the other two will violate the two independent boundary conditions (21b) and (21e) on H_0 and $H_{0,r}$. Only for particular (usually discrete) values of the frequency, ω—the star's "eigenfrequencies"—will a linear combination of the two acceptable surface solutions match onto a linear combination of the two acceptable central solutions.

The eigenfrequencies and eigenfunctions can be calculated for any stellar model by a series of trial-and-error integrations. In each trial integration one can: (i) pick a trial frequency ω, (ii) integrate the two acceptable central solutions out from $r = 0$ to $r \sim \frac{1}{2}R$ ("match point"), (iii) integrate the two acceptable surface solutions inward from $r = R$ to the match point, (iv) try to match a linear combination of the surface solutions to a linear combination of the central solutions.

Because no provision was made in our stress-energy tensor for dissipation, and because no dipole gravitational waves are possible, each solution of the eigenvalue problem should either pulsate sinusoidally with no losses (ω real) or grow exponentially with no oscillations (ω imaginary). It would be interesting to prove this conjecture from the eigenequations directly—i.e., to prove that, for all solutions that satisfy our boundary conditions, ω^2 is real.

IV. CONCLUSION

This completes our series of five papers on the theory of the nonradial pulsation of relativistic stellar models. Since this series was begun, the discovery of pulsars has made this work much more relevant for astrophysics than it was originally. At the same time, however, pulsar research has called into question one of the fundamental premises on which this work rests: the assumption that the stellar material can be idealized as a perfect fluid. At subnuclear densities, according to Ruderman (1968, 1969), neutron-star matter might crystalize and might support shear stresses. If so, then the theory developed in this series of papers should be extended to stars with fluid cores and crystalline mantles.

We thank Barbara Zimmerman for assistance with the computer calculations of the Einstein tensor and of the divergence of the stress-energy tensor. One of us (K. S. T.) thanks the Alfred P. Sloan Foundation for a Research Fellowship during part of the period of this research.

APPENDIX A

ODD-PARITY PERTURBATIONS

Odd-parity perturbations for $l \geq 2$ were treated in Appendices A and B of Paper I. The analysis for $l = 1$ differs from that for $l \geq 2$ because the odd-parity tensorial spherical harmonic, $\chi^1{}_{Mjk}$, vanishes identically.

In a *general* coordinate system which agrees, before perturbation, with the Schwarzschild coordinates of the star (cf. eq. [I,1]), a *general* odd-parity, $l = 1$ perturbation has the form of equation (I,A5)—but with $h_{jk} = 0$ because $\chi^1{}_{Mjk} \equiv 0$:

$$\xi_r = 0\,, \qquad \xi_j = U(r, t)\Phi^1{}_{Mj}\,;$$
$$h_{00} = h_{0r} = h_{rr} = h_{jk} = 0\,, \qquad h_{0j} = h_0(r, t)\Phi^1{}_{Mj}\,, \qquad h_{rj} = h_1(r, t)\Phi^1{}_{Mj}\,. \tag{A1}$$

The expression for the metric perturbation can be simplified by a specialization of the gauge:

$$x^{\alpha'} = x^\alpha + \eta^\alpha\,, \qquad \eta_0 = \eta_r = 0\,, \qquad \eta_j = \Lambda(r, t)\Phi^1{}_{Mj}\,,$$
$$h_{\alpha\beta}' = h_{\alpha\beta} - (\eta_{\alpha;\beta} + \eta_{\beta;\alpha})\,. \tag{A2}$$

In Paper I $\Lambda(r, t)$ was chosen so as to make h_{jk}' vanish. However, for $l = 1$, h_{jk}', like h_{jk}, will vanish automatically because $\chi^1{}_{Mjk} = 0$. Consequently, we can now choose $\Lambda(r, t)$ so as to make h_{rj}' vanish. By doing this, we obtain a gauge in which (with primes omitted henceforth, and with a change in notation)

$$\xi^r = 0\,, \quad \xi^j = -(4\pi/3)^{1/2}\left[\int_0^t \Omega(r, t)dt\right]\Phi^1{}_M{}^j\,, \tag{A3}$$

$$h_{0j} = -(4\pi/3)^{1/2} r^2 \omega(r, t)\Phi^1{}_{Mj}, \; h_{00} = h_{0r} = h_{rr} = h_{rj} = h_{jk} = 0\,.$$

Here, and always, the tensor indices on spherical harmonics are raised and lowered by using the two-sphere metric—$\gamma_{\theta\theta} = 1$, $\gamma_{\theta\phi} = 0$, $\gamma_{\phi\phi} = \sin^2\theta$—rather than the metric of spacetime, and indices j and k run over θ and ϕ (i.e., 2 and 3).

The fluid four-velocity corresponding to equation (A3) is

$$u^0 = e^{-\nu/2}\,, \quad u^j = -(4\pi/3)^{1/2}\Omega e^{-\nu/2}\Phi^1{}_M{}^j\,, \quad u^r = 0\,. \tag{A4}$$

The Lagrangian and Eulerian perturbations in pressure and density are zero because their angular distributions must be described by scalar spherical harmonics with $l = 1$ and $\pi = (-1)^{l+1} = +1$, and no such scalar spherical harmonics exist. Consequently, the Eulerian perturbation in the stress-energy tensor is $\delta T_\mu{}^\nu = (\rho + p)\,(u_\mu \delta u^\nu + u^\nu \delta u_\mu)$. This has as its only nonzero components

$$\delta T_j{}^0 = r^2 e^{-\nu}(\rho + p)(\Omega - \omega)(4\pi/3)^{1/2}\Phi^1{}_{Mj} \tag{A5}$$

and $\delta T_0{}^j$, which is readily calculated but is not needed here. The perturbation in the Einstein tensor, associated with the metric perturbation of equation (A3), has the following nonzero components:

$$\delta G_j{}^0 = -\tfrac{1}{2} r^{-2} e^{-(\nu+\lambda)/2}[e^{-(\nu+\lambda)/2} r^4 \omega_{,r}]_{,r}(4\pi/3)^{1/2}\Phi^1{}_{Mj}\,, \tag{A6a}$$

$$\delta G_j{}^r = \tfrac{1}{2} r^2 e^{-\lambda-\nu}\omega_{,rt}(4\pi/3)^{1/2}\Phi^1{}_{Mj}\,; \tag{A6b}$$

$\delta G_0{}^j$ and $\delta G_r{}^j$ are also nonzero, but we do not need them since they are not independent of $\delta G_j{}^0$ and $\delta G_j{}^r$. By constructing Einstein's equations, $\delta G_\alpha{}^\beta = 8\pi\delta T_\alpha{}^\beta$, from equations (A5) and (A6), by manipulating them, by combining them with the zero-order field equations (I, 2 and 3), and by demanding that ω approach zero as r approaches infinity, we obtain

$$\Omega_{,t} = \omega_{,t} = 0\,, \tag{A7a}$$

$$r^{-4}[r^4 e^{-(\lambda+\nu)/2}\omega_{,r}]_{,r} + 4r^{-1}[e^{-(\lambda+\nu)/2}]_{,r}(\omega - \Omega) = 0\,. \tag{A7b}$$

The physical interpretations of the perturbations (A3) and corresponding field equations (A7) are given in the text.

APPENDIX B

METRIC AND FLUID PERTURBATIONS FOR EVEN PARITY

Even-parity perturbations for $l \geq 2$ were treated in Appendix A of Paper I. The analysis for $l = 1$ differs from that for $l \geq 2$ because the even-parity tensorial harmonics satisfy the identity $\Phi^1{}_{Mjk} = -\Psi^1{}_{Mjk}$.

Before specialization of the gauge, the general $l = 1$, even-parity perturbation has the form of equation (I,A6), with $\Phi^1{}_{Mjk}$ and $\Psi^1{}_{Mjk}$ interchanged in h_{jk} (note the errata to Paper I):

$$\begin{aligned}
\xi_r &= X(r, t)\,Y^1{}_M\,, \quad \xi_j = V(r, t)\Psi^1{}_{Mj}\,;\\
h_{00} &= e^\nu H_0(r, t)\,Y^1{}_M\,, \quad h_{0r} = H_1(r, t)\,Y^1{}_M\,, \quad h_{rr} = e^\lambda H_2(r, t)\,Y^1{}_M\,;\\
h_{0j} &= h_0(r, t)\Psi^1{}_{Mj}\,, \quad h_{rj} = h_1(r, t)\Psi^1{}_{Mj}\,:\\
h_{jk} &= r^2[G(r, t) - K(r, t)]\Psi^1{}_{Mjk}\,.
\end{aligned} \tag{B1}$$

The expression for the metric perturbation can be simplified by a specialization of the gauge:

$$x^{\alpha\prime} = x^{\alpha} + \eta^{\alpha}\,, \quad \eta_0 = M_0(r, t)\, Y^1{}_M\,, \quad \eta_r = M_1(r, t)\, Y^1{}_M\,, \quad \eta_j = M_2(r, t)\Psi^1{}_{Mj}\,. \tag{B2}$$

In Paper I we chose M_0, M_1, and M_2 so as to annul the functions h_0, h_1, and G. Here, because of our simplified form for h_{jk}, we can annul K at the same time as we annul G—i.e., we can annul $G - K$, the only combination in which G and K appear. By doing this, we obtain a gauge in which (with primes omitted and with a change in notation)

$$\begin{aligned} &\xi_r = -r^{-2}e^{\lambda/2}W(r, t)\, Y^1{}_M\,, \quad \xi_j = V(r, t)\Psi^1{}_{Mj}\,; \\ &h_{00} = e^{\nu}H_0(r, t)\, Y^1{}_M\,, \quad h_{0r} = H_1(r, t)\, Y^1{}_M\,, \quad h_{rr} = e^{\lambda}H_2(r, t)\, Y^1{}_M\,, \\ &h_{0j} = h_{rj} = h_{jk} = 0\,. \end{aligned} \tag{B3}$$

APPENDIX C

NONUNIQUENESS OF EVEN-PARITY GAUGE

The $l = 1$ gauge of equation (9) (or equivalently eq. [B3]), and the $l \geq 2$ Regge-Wheeler gauge of Paper I are specified completely by the conditions

$$\begin{aligned} &h_{00} = e^{\nu}H_0(r, t)\, Y^l{}_M\,, \quad h_{0r} = H_1(r, t)\, Y^l{}_M\,, \quad h_{rr} = e^{\lambda}H_2(r, t)\, Y^l{}_M\,, \\ &h_{0j} = h_{rj} = 0\,, \quad h_{jk} = r^2K(r, t)\Phi^l{}_{Mjk}\,, \\ &K(r, t) = 0 \quad \text{if} \quad l = 1\,. \end{aligned} \tag{C1}$$

Is there a change of gauge which leaves the form of these perturbations unchanged?

To answer this question, consider the most general change of gauge which does not produce spherical harmonics with other values of l, M, *and parity:*

$$\begin{aligned} &x^{\alpha\prime} = x^{\alpha} + \eta^{\alpha}\,, \quad \eta_0 = M_0(r, t)\, Y^l{}_M\,, \quad \eta_r = M_1(r, t)\, Y^l{}_M\,, \quad \eta_j = M_2(r, t)\Psi^l{}_{Mj}\,, \\ &h_{\alpha'\beta'} = h_{\alpha\beta} - (\eta_{\alpha;\beta} + \eta_{\beta;\alpha})\,. \end{aligned} \tag{C2}$$

Straightforward calculation yields

$$\begin{aligned} h_{0'0'} &= [e^{\nu}H_0 - 2M_{0,t} + \nu_{,r}e^{\nu-\lambda}M_1]\, Y^l{}_M\,, \\ h_{0'r'} &= [H_1 - M_{0,r} + \nu_{,r}M_0 - M_{1,t}]\, Y^l{}_M\,, \\ h_{r'r'} &= [e^{\lambda}H_2 - 2M_{1,r} + \lambda_{,r}M_1]\, Y^l{}_M\,, \\ h_{0'j'} &= -[M_{2,t} + M_0]\Psi^l{}_{Mj}\,, \\ h_{r'j'} &= -[M_{2,r} - 2r^{-1}M_2 + M_1]\Psi^l{}_{Mj}\,, \\ h_{j'k'} &= [r^2K - 2re^{-\lambda}M_1]\Phi^l{}_{Mjk} - 2M_2\Psi^l{}_{Mjk}\,. \end{aligned} \tag{C3}$$

For $l \geq 2$, when $\Phi^l{}_{Mjk}$ and $\Psi^l{}_{Mjk}$ are completely independent functions, the new gauge will have the same form as the old one if and only if

$$M_{2,t} + M_0 = 0\,, \quad M_{2,r} - 2r^{-1}M_2 + M_1 = 0\,, \quad M_2 = 0\,. \tag{C4}$$

The most general solution of these equations is $M_0 = M_1 = M_2 = 0$. Hence, for $l \geq 2$ the gauge of equation (C1) is unique.

For $l = 1$, when $\Phi^l{}_{Mjk} = -\Psi^l{}_{Mjk}$, the new gauge will have the same form as the old one if and only if

$$M_{2,t} + M_0 = 0\ , \qquad M_{2,r} - 2r^{-1}M_2 + M_1 = 0\ , \qquad 2re^{-\lambda}M_1 - 2M_2 = 0\ . \tag{C5}$$

The general solution to these coupled equations is readily shown to be

$$M_0 = -a_{,t}f\ , \quad M_1 = ar^{-1}e^{\lambda}f\ , \quad M_2 = af\ ,$$

$$f = r\exp\left[-\int_r^{\infty} r^{-1}(1 - e^{\lambda})dr\right]\ , \quad a = a(t) = \text{arbitrary function of } t\ . \tag{C6}$$

Hence, for $l = 1$ the gauge of equation (C1) is unique only up to transformations of the form (C2), (C6). The changes in the perturbation functions, H_0, H_1, H_2, W, and V, produced by such transformations, are readily calculated by combining equations (C1)–(C3), (C6), and (10). The results are given in equations (13).

APPENDIX D

EQUATIONS OF MOTION FOR EVEN PARITY

For the even-parity perturbations of equations (9) and (10), the perturbed Einstein field equations are calculated by the same method as we used in Appendix C of Paper I. The perturbation in the stress-energy tensor is identical with that of Paper I, except that the angular dependence is specialized to $l = 1$, M not necessarily zero, and the function K vanishes:

$$\delta T_r{}^r = \delta T_\theta{}^\theta = \delta T_\phi{}^\phi = [e^{-\nu/2}S + r^{-2}e^{-\lambda/2}p_{,r}W]Y^1{}_M\ , \tag{D1a}$$

$$\delta T_0{}^0 = [-(\rho + p)(\gamma p)^{-1}e^{-\nu/2}S - r^{-2}e^{-\lambda/2}\rho_{,r}W]Y^1{}_M\ , \tag{D1b}$$

$$\delta T_0{}^r = (\rho + p)r^{-2}e^{-\lambda/2}W_{,t}Y^1{}_M\ , \tag{D1c}$$

$$\delta T_0{}^j = -(\rho + p)r^{-2}V_{,t}\Psi^1{}_M{}^j\ . \tag{D1d}$$

The only other nonvanishing components, $\delta T_r{}^0$ and $\delta T_j{}^0$, are dependent on these. In these expressions S is the function defined in equation (11). Notice that $-Se^{-\nu/2}Y^1{}_M$ is the Lagrangian change in pressure.

The perturbation in the Einstein tensor, as calculated by using the computer programs ALBERT (cf. Thorne and Zimmerman 1967) has components

$$\delta G_r{}^r = [2r^{-1}e^{-\lambda-\nu}H_{1,t} - r^{-1}e^{-\lambda}H_{0,r} + r^{-2}H_0 - r^{-1}(r^{-1} + \nu_{,r})e^{-\lambda}H_2]Y^1{}_M\ , \tag{D2a}$$

$$\begin{aligned}\delta G_\theta{}^\theta = \delta G_\phi{}^\phi = \{&-\tfrac{1}{2}e^{-\nu}H_{2,tt} + e^{-\lambda-\nu}H_{1,tr} + (r^{-1} - \tfrac{1}{2}\lambda_{,r})e^{-\lambda-\nu}H_{1,t} - \tfrac{1}{2}e^{-\lambda}H_{0,rr}\\ &- \tfrac{1}{2}e^{-\lambda}(\nu_{,r} - \tfrac{1}{2}\lambda_{,r} + r^{-1})H_{0,r} + \tfrac{1}{2}r^{-2}H_0 - \tfrac{1}{2}e^{-\lambda}(r^{-1} + \tfrac{1}{2}\nu_{,r})H_{2,r}\\ &+ \tfrac{1}{2}e^{-\lambda}[-r^{-2}e^{\lambda} + r^{-1}(\lambda_{,r} - \nu_{,r}) + \tfrac{1}{2}\lambda_{,r}\nu_{,r} - \tfrac{1}{2}\nu_{,r}{}^2 - \nu_{,rr}]H_2\}Y^1{}_M\ ,\end{aligned} \tag{D2b}$$

$$\delta G_0{}^0 = \{-r^{-1}e^{-\lambda}H_{2,r} + r^{-1}[e^{-\lambda}\lambda_{,r} - r^{-1}(1 + e^{-\lambda})]H_2\}Y^1{}_M\ , \tag{D2c}$$

$$\delta G_0{}^r = (r^{-1}e^{-\lambda}H_{2,t} + r^{-2}e^{-\lambda}H_1)Y^1{}_M\ , \tag{D2d}$$

$$\delta G_0{}^j = -\tfrac{1}{2}r^{-2}[H_{2,t} - e^{-\lambda}H_{1,r} + \tfrac{1}{2}e^{-\lambda}(\lambda_{,r} - \nu_{,r})H_1]\Psi^1{}_M{}^j\ , \tag{D2e}$$

$$\delta G_r{}^j = -\tfrac{1}{2}r^{-2}[e^{-\nu}H_{1,t} - H_{0,r} + (r^{-1} - \tfrac{1}{2}\nu_{,r})H_0 - (r^{-1} + \tfrac{1}{2}\nu_{,r})H_2]\Psi^1{}_M{}^j\ . \tag{D2f}$$

The only other nonvanishing components, $\delta G_r{}^0$, $\delta G_j{}^0$, and $\delta G_j{}^r$, are dependent on these.

The divergence of the stress-energy tensor for the perturbed star has as its perturbations

$$\delta(T_r{}^{\mu}{}_{;\mu}) = \{-r^{-2}(\rho + p)e^{\lambda/2-\nu}W_{,tt} + e^{-\nu/2}S_{,r}$$
$$+ [p_{,rr} + \tfrac{1}{2}(\rho + p)_{,r}\nu_{,r} - (2r^{-1} + \tfrac{1}{2}\lambda_{,r})p_{,r}]r^{-2}e^{-\lambda/2}W \qquad \text{(D3a)}$$
$$+ (\rho + p)\nu_{,r}r^{-2}V - \tfrac{1}{4}(\rho + p)\nu_{,r}H_2 - \tfrac{1}{2}(\rho + p)H_{0,r} + (\rho + p)e^{-\nu}H_{1,t}\} Y^1{}_M$$

$$\delta(T_j{}^{\mu}{}_{;\mu}) = \{(\rho + p)e^{-\nu}V_{,tt} + e^{-\nu/2}S + r^{-2}e^{-\lambda/2}p_{,r}W - \tfrac{1}{2}(\rho + p)H_0\}\Psi^1{}_{Mj}\,. \qquad \text{(D3b)}$$

$$\delta(T_0{}^{\mu}{}_{;\mu}) \equiv 0\,. \qquad \text{(D3c)}$$

(The procedure used to construct the stress-energy tensor [Appendix C of Paper I], together with $T_\mu{}^\nu{}_{;\nu} = 0$ for the unperturbed star, guarantees that $\delta(T_0{}^{\mu}{}_{;\mu}) \equiv 0$.)

From the above expressions for $\delta T_\mu{}^\nu$, $\delta G_\mu{}^\nu$, and $\delta(T_\mu{}^\nu{}_{;\nu})$ one can readily obtain the field equations presented in § III*c*: The equation $\delta G_r{}^j = 8\pi\delta T_r{}^j = 0$ is an initial-value equation for $H_{1,t}$:

$$H_{1,t} = e^\nu H_{0,r} + (\tfrac{1}{2}\nu_{,r} - r^{-1})e^\nu H_0 + (\tfrac{1}{2}\nu_{,r} + r^{-1})e^\nu H_2\,. \qquad \text{(D4)}$$

By using this equation to eliminate $H_{1,t}$ from $\delta(T_r{}^{\mu}{}_{;\mu}) = 0$ (eq. [D3*a*]), and by using the zero-order field equations (eqs. [I,1]–[I,3]) to simplify some of the terms, one obtains the dynamical equation (14a) for W. The equation $\delta(T_j{}^{\mu}{}_{;\mu}) = 0$ (eq. [D3b]) becomes the dynamical equation (14b) for V without any manipulation. By combining the equation $\delta G_0{}^0 = 8\pi\delta T_0{}^0$ (eqs. [D2c] and [D1b]) with the definition (11) of S, one obtains the initial-value equation (15a) for H_2. Equation (15b) is merely a restatement of the definition (11) of S. By using equation (D4) to eliminate $H_{1,t}$ from $\delta G_r{}^r = 8\pi\delta T_r{}^r$ (eqs. [D2a] and [D1a]), one obtains the initial-value equation (15c) for H_0. The equation $\delta G_0{}^r = 8\pi\delta T_0{}^r$ (eqs. [D2d] and [D1c]) becomes the initial-value equation (15d) for H_1, without manipulation; and equation (D4) becomes the initial-value equation (15d′) for $H_{1,t}$.

One can verify that equations (14) and (15) are a complete set of field equations, i.e., that, if they are satisfied, then all of the equations $\delta G_\mu{}^\nu = 8\pi\delta T_\mu{}^\nu$ and $\delta(T_\mu{}^\nu{}_{;\nu}) = 0$ are satisfied.

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