ASTROPHYSICAL RELATIVITY*

(Relativistic Stars and Star Clusters: Their Structure, Rotation, Pulsation, and Emission of Gravitational Waves)

KIP S. THORNE†

California Institute of Technology, Pasadena, California USA

and

University of Chicago, Chicago, Illinois USA

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I. INTRODUCTION

Recent astronomical discoveries and calculations have stimulated wide interest in the astrophysical effects of strong (relativistic) gravitational fields. Previous to about 1962 the view prevailed that general relativistic effects were significant only for cosmology; that in isolated astronomical bodies general relativity is probably unimportant. Since 1962, however, one has come to believe that strong gravitational fields may be of key importance for quasars, for violent events in the nuclei of galaxies, for supernova explosions and remnants, for the death-by-collapse of very massive stars, and for the periodic-burst radio sources of Hewish et al. (1968). (Reviews will be found in Zel'dovich and Novikov [1964; 1965; 1967]; Harrison, Thorne, Wakano, Wheeler [1965]; Wheeler [1966]; Thorne [1967 a,b].)

In order to facilitate theoretical investigations of these phenomena, considerable effort has been devoted, in the last few years, to the development of the theory of relativistic stars and star clusters, including: their structure, stability, rotation, pulsation, collapse, and emission of gravitational waves. In this lecture I shall review recent theoretical studies of these topics - - - omitting, however, the topic of collapse, which undoubtedly will be discussed in detail by others at this conference. The emphasis of this review will be on general relativity; astrophysical applications will be relegated to a subordinate role.

II. RELATIVISTIC STELLAR MODELS

Relativistic effects are believed to be important in two types of stars: Supermassive stars ($M \gtrsim 10^4 M_\odot$), which may be energy sources for quasars and galactic nuclei (Hoyle and Fowler 1963, 1965); and stars at the endpoint of thermonuclear evolution (neutron stars and collapsed stars).

A. Equilibrium Configurations

The equations of structure for relativistic stellar models were first derived and used by Oppenheimer and Volkoff (1939). (For recent reviews see Thorne [1967 a,b], Zel'dovich and Novikov [1967]). The gravitational field of a nonrotating, relativistic star is described by the line element

$$ds^2 = e^{\nu} dt^2 - (1 - 2\, m/r)^{-1}\, dr^2 - r^2 (d\theta^2 + \sin^2\theta\, d\Phi^2), \tag{1}$$

where m and ν are functions of r. (We set c = G = 1 throughout.) The stellar material is idealized as a perfect fluid, at rest in the coordinate system (1), with pressure, p, and density of mass-energy, ρ. The equations of structure for the star consist of an equation of state

$$p = p\,(\rho), \tag{2a}$$

Einstein's field equations

$$dm/dr = 4\pi r^2 \rho, \tag{2b}$$

$$\frac{dp}{dr} = - \frac{(\rho + p)(m + 4\pi r^3 p)}{r(r-2m)} \quad , \tag{2c}$$

$$d\nu/dr = - 2(\rho + p)^{-1}(dp/dr) \quad , \tag{2d}$$

and boundary conditions

$$m(r = 0) = 0, \quad \nu(r = \infty) = 0\ , \quad p(r = \text{surface of star}) = 0. \tag{2e}$$

When energy transport (conduction, convection, radiation) is taken into account these equations must be augmented and modified (cf. Thorne 1967 a,b). On the other hand, when energy transport is ignored, equations (2) yield a unique stellar model for each choice of the equation of state, $p(\rho)$, and central density, $\rho(r = 0) = \rho_c$.

Since 1957 a large number of relativistic stellar models have been constructed to represent neutron stars and supermassive stars; and a number of theorems have been proved about the properties of relativistic stellar models. A key result of astrophysical significance is this: There exists no equilibrium configuration at the endpoint of thermonuclear evolution with mass greater than about 2 solar masses. Any star more massive than this must eject its excess mass or face eventual death by collapse through the gravitational radius.

B. Stability of Equilibrium Configurations

In 1963 S. Chandrasekhar (1964) and R. P. Feynman (unpublished work which greatly influenced the Hoyle-Fowler ideas on supermassive stars) discovered a relativistic effect of extreme importance for astrophysics:[1]

[1] This effect had previously been noticed by Kaplan (1949); but his work went virtually unnoticed and had no influence on astrophysical thinking.

They developed independently the mathematical theory of radial pulsations of relativistic stellar models, and in the process they discovered that general relativity can catalyze an instability in stars, which is absent in Newtonian theory: In Newtonian theory a star is stable against radial perturbations if and only if its adiabatic index, suitably averaged over the stellar interior, is greater than 4/3. However, relativistic effects raise the critical value of the adiabatic index above 4/3:

$$\Gamma_{crit} = 4/3 + \text{relativistic term of } O(M/R). \tag{3}$$

As a consequence, a star, in which relativistic influence on the structure is negligible, and which is just barely stable according to Newtonian theory ($<\Gamma>$ only slightly greater than 4/3), may actually be unstable because of relativistic effects. Such is the case, for example, in a nonrotating supermassive star of 10^8 solar masses with $M/R \approx 10^{-4}$.

C. Nonradial Pulsation and Gravitational Waves

The nonradial pulsations of relativistic stellar models were first investigated by Chandrasekhar (1965b) using the post-Newtonian approximation, in which there are no gravitational waves. More recently Thorne and Campolattaro (1967), Price and Thorne (1968), and Thorne (1968) have studied nonradial pulsations within the framework of the full, nonlinear theory of general relativity. Using techniques commonplace in nuclear physics, particle physics, and quantum theory, they have analyzed the manner in which pulsating stars emit gravitational waves, and the damping effects of the gravitational waves on the stellar pulsations. Because this analysis provides the most rigorous example yet given of the coupling of gravitational waves to their source, I present here a brief outline of the analysis:

The stellar material is treated as a perfect fluid, which is displaced by an amount $\underset{\sim}{\xi}$ (t,r,θ, φ) relative to the coordinate system. The gravitational field is characterized by the unperturbed metric (1) plus a perturbation, $h_{\mu\nu}(t, r, \theta, \varphi)$. A particular choice of guage (i.e., coordinate system) is made; and a computer is used (see Fletcher <u>et al</u>. 1967) to derive the explicit, analytic form for Einstein's field equations in that guage. The field equations are linearized in $\underset{\sim}{\xi}$ and $h_{\mu\nu}$, but no other approximation is made. (Note that the equilibrium configuration about which one perturbs is fully relativistic!)

Attention is concentrated initially on "standing-wave normal modes", in which the star pulsates sinusoidally and standing gravitational waves couple it to a perfect reflector at $r = \infty$. Each standing-wave normal mode is characterized by spherical-harmonic indices (ℓ, m) and by the pulsation angular frequency, ω. If the frequency, ω, is very near a natural pulsation frequency, σ, for the star; then the star's pulsation energy, E_M, is large compared to the energy, E_W, in one wavelength of the gravitational waves. (E_W is calculated using the Landau-Lifschitz pseudotensor.) As ω is allowed to vary near σ, the ratio of energy in matter to energy in waves goes through a resonance

$$\frac{E_M}{E_W} \propto \frac{1}{(\omega-\sigma)^2 + (1/\tau)^2} \quad . \tag{4}$$

(cf. Thorne 1968 for proof.) From the physical viewpoint this resonance is associated with the natural mode of vibration of the star at $\omega = \sigma$; from the mathematical viewpoint it is associated with a pole in the S-matrix for scattering of gravitational waves off the star, which pole is located at

$$\omega_{\text{pole}} = \sigma + i/\tau \tag{5}$$

in the complex frequency plane. Thorne (1968) has explored numerically the resonances for a variety of neutron-star models. (See Figure 1.)

The above discussion refers to the unrealistic problem of standing-wave modes. A more realistic situation is obtained by building a "wave packet" of standing waves peaked about a resonant frequency, σ. For example, the wave packet.

$$\left\{ h_{\mu\nu}, \xi \right\} = \int \frac{\left\{ h_{(\omega)\mu\nu}, \xi_{(\omega)} \right\} e^{i\omega t}}{(\omega-\sigma)^2 + (1/\tau)^2} \, d\omega \tag{6}$$

represents a star in which gravitational waves flow in from $r = \infty$, exciting the star into pulsation; and then the star reradiates the waves. (See Figure 2.) The total energy carried in and back out by the waves is finite and can be arbitrarily small, so the solution (6) should be an excellent approximation to an exact solution of Einstein's field equations.

A close examination of the solution (6) reveals this, that as the star reradiates its gravitational waves, the amplitude of its pulsations (and hence also of the waves) is damped exponentially,

$$\text{Amplitude} \propto e^{-t/\tau}, \tag{7}$$

with a damping rate, $1/\tau$, equal to the half-width of the standing-wave resonance. An analytic form for the reradiated waves in the radiation zone is

$$h_{\alpha\beta} = 0 \quad \text{if } t < r + 2M \, \ln(r-2M);$$

$$\frac{h_{\theta\theta}}{r^2} = -\frac{h_{\varphi\varphi}}{r^2\sin^2\theta} = A\left[\frac{\ell(\ell+1)}{2}\, P_\ell^m(\cos\theta) + \frac{\partial^2 P_\ell^m(\cos\theta)}{\partial\theta^2}\right]\frac{1}{r}\exp\left[-\frac{t-r-2M\,\ln(r-2M)}{\tau}\right]$$

$$\times \sin\{\sigma\,[t-r-2M\,\ln(r-2M)] + m\varphi + \delta\} + O(1/r^2),$$

$$\frac{h_{\theta\varphi}}{r^2\sin\theta} = Am\left[\frac{1}{\sin\theta}\,\frac{\partial P_\ell^m(\cos\theta)}{\partial\theta} - \frac{\cos\theta}{\sin^2\theta}\,P_\ell^m(\cos\theta)\right]\frac{1}{r}\exp\left[-\frac{t-r-2M\,\ln(r-2M)}{\tau}\right]$$

$$\times \cos\{\sigma\,[t-r-2M\,\ln(r-2M)] + m\varphi + \delta\} + O(1/r^2)$$

$$\text{if } t > r + 2M\,\ln(r-2M). \tag{8}$$

Here $h_{\alpha\beta}$ is the perturbation of the Schwarzschild metric, and A and δ are constants. Notice that far from the star these waves are very nearly identical to the plane, weak waves of linearized general relativity (Landau and Lifschitz 1962, § 101). They exhibit linear polarization if $\mathcal{m} = 0$, and elliptical polarization if $\mathcal{m} \neq 0$.

The pulsation angular frequencies, σ, and damping time, τ, have been calculated by Thorne (1968) for the quadrupole pulsations of a variety of neutron-star models. The pulsation periods for neutron stars of mass $M \gtrsim 0.5\ M_{\odot}$ are typically

$$T = 2\pi/\sigma \sim 0.1 \text{ to } 0.3 \text{ milliseconds;} \tag{9}$$

and the damping times are typically

$$\tau \sim 0.1 \text{ to } 10 \text{ seconds.} \tag{10}$$

Consequently, immediately after a neutron star is formed in a supernova explosion, it should emit $\sim 10^{52}$ ergs in a burst of gravitational waves in the kilocycle frequency region; and the burst should last for only about 1 second.

D. Rotation

The effects of rotation upon the structure and stability of relativistic stellar models have been investigated recently using two different approximations: (1) The post-Newtonian approximation has been used to examine small relativistic effects in slowly and rapidly rotating stars (Chandrasekhar 1965 a, 1967 a, b, c; Fowler 1966, Durney and Roxburgh 1967; Bisnovatny-Kogan, Zel'dovich, and Novikov 1967); and (2) The small-rotation approximation has been used to examine the effects of rotation on fully relativistic stars (Hartle 1967; Hartle and Thorne 1968; Chitre, Hartle, and Thorne 1968).

Several new effects emerge when rotation and relativity are coupled:

(1) Inertial frames are "dragged" by the rotating star. This well-known "Lense-Thirring effect" has been studied extensively in the above papers and also by Brill and Cohen (1966). (2) Rotation helps to stabilize stellar models; for example, it can delay the onset of the relativistic instability in supermassive stars. (3) Rotation deforms a star and thereby couples its radial modes of pulsation to its quadrupole modes. That coupling, and the resultant emission of gravitational waves by the quasi-radial pulsations have been investigated by Chitre, Hartle, and Thorne (1968), using techniques analogous to those described in the last section.

These effects of rotation will be discussed in some detail in the lecture.

III. RELATIVISTIC STAR CLUSTERS

Relativistic star clusters, like relativistic stars, may play an important role in high-energy astrophysics: Hoyle and Fowler (1967) have suggested a model for quasars in which the compact source of optical radiation resides at the center of a very massive ($M \sim 10^{13}\ M_\odot$) cluster of stars, and derives most of its redshift ($z \sim 0.2$ to 2.2) from the cluster's relativistic gravitational field. Other investigations (see Ipser and Thorne 1968 for references) have suggested that relativistic clusters might evolve in the nuclei of some galaxies and might be associated with the explosions which disrupt galactic nuclei.

One expects that relativistic star clusters should exhibit properties very similar to those of relativistic stars (e.g., instability against collapse when they become too compact; emission of gravitational waves if they are not stationary; deformation and dragging of inertial frames when they rotate. However, the development of the theory of relativistic star clusters has begun only recently: Spherically symmetric equilibrium configurations for star clusters have been studied by Zel'dovich and Podurets (1965) and by Fackerell (1968a, b). Fackerell (1968c) has initiated the study of rotating star clusters. The theory of the spherical pulsations of spherical clusters has been developed by Ipser and Thorne (1968) and is now being used by Ipser to study the onset of the relativistic instability in clusters.

In the lecture I will describe these analyses of star clusters and their implications for quasars and galactic nuclei.

IV. CONCLUSION

The structures of nonrotating stars, the effects of rotation upon them, their small-amplitude pulsations, and the resultant emission of gravitational waves --- all of these things are now moderately well understood within the framework of general relativity when either (i) relativistic effects are small (post-Newtonian approximation), or (ii) the departures from spherical symmetry are small (small-amplitude approximation). These were the easy problems. The hard --- and more interesting --- problem which remains is that of generalizing these analyses to the large-amplitude, fully relativistic situation.

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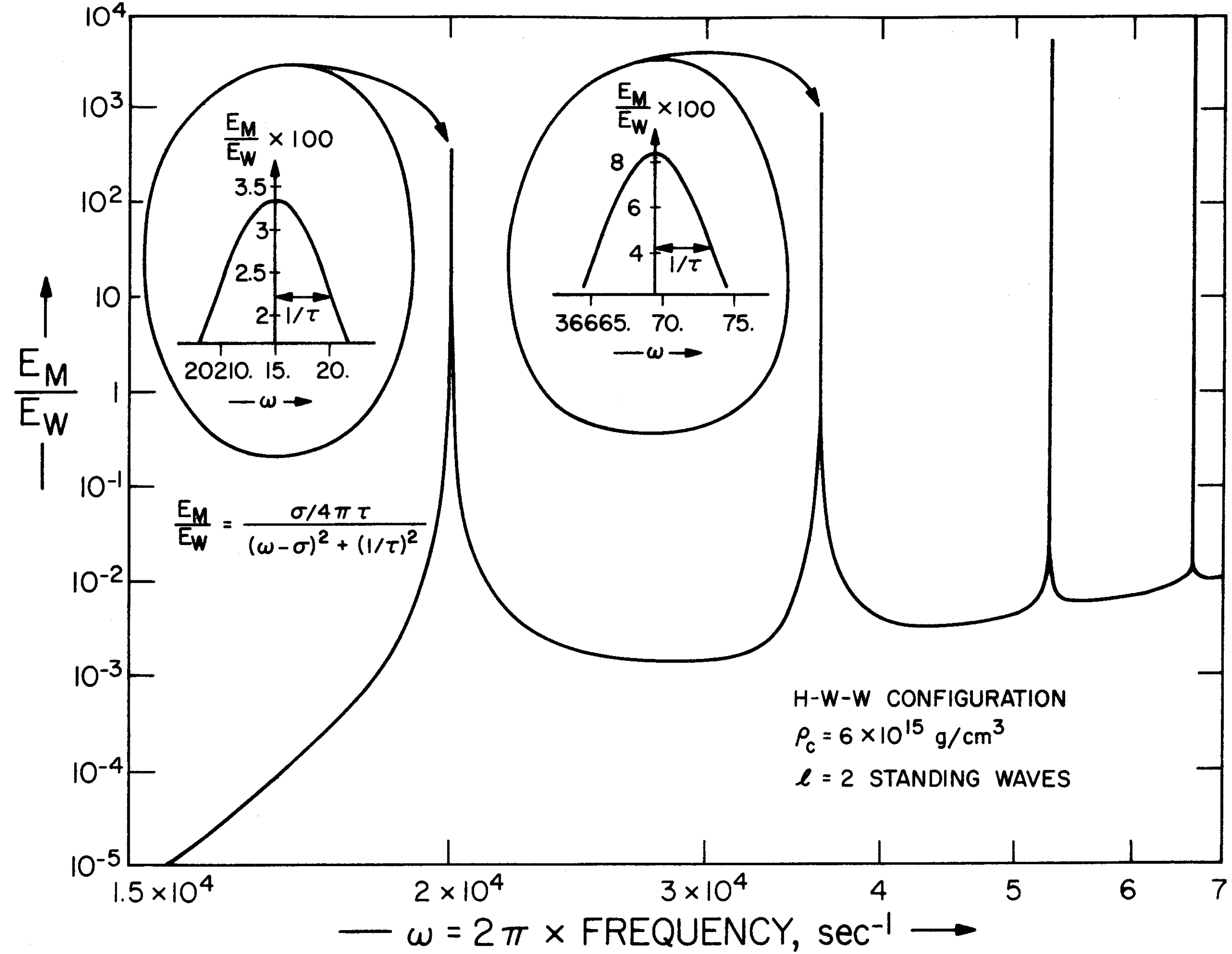
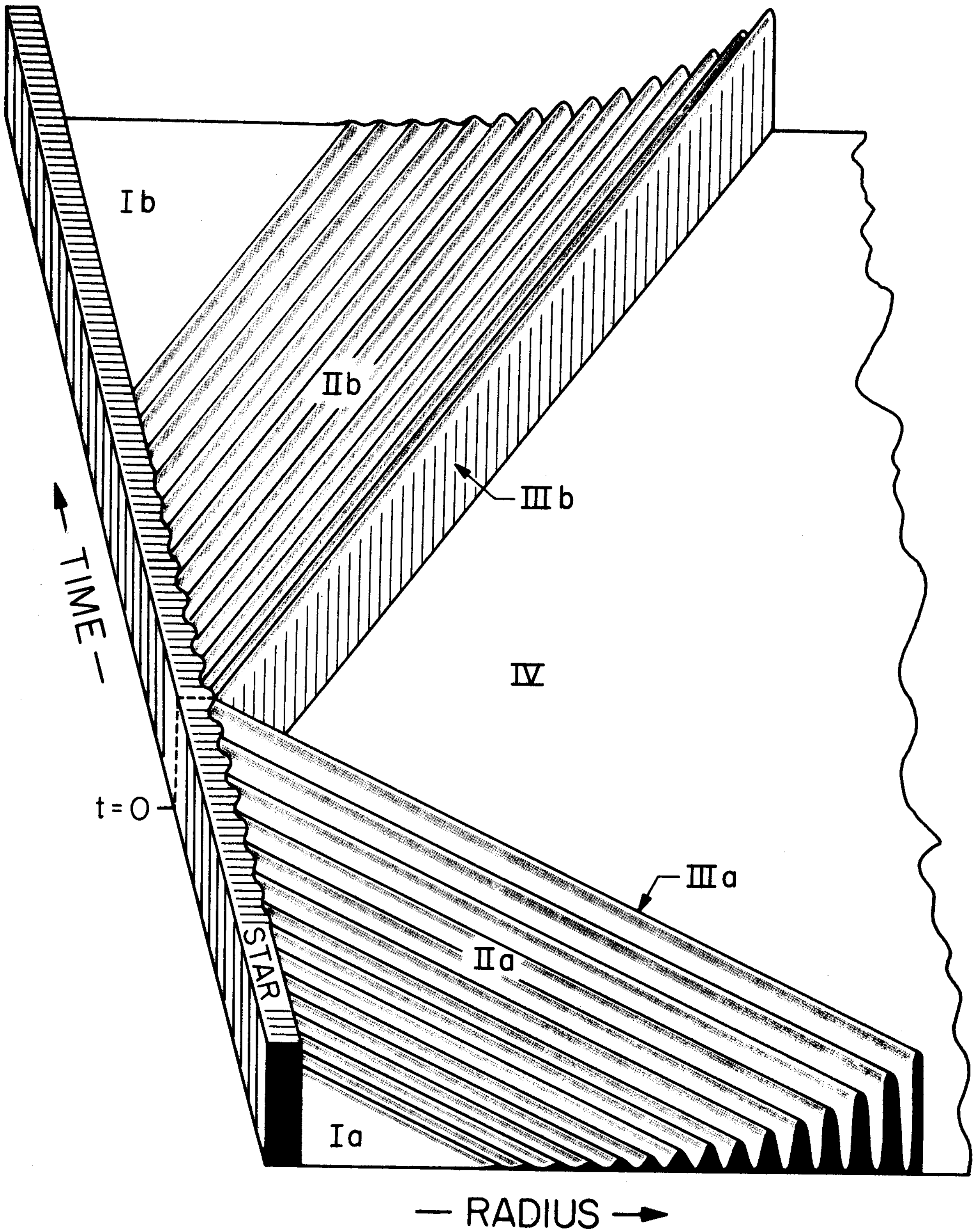
FIGURE CAPTIONS

Figure 1 Resonances in the quadrupole standing-wave normal modes for a H-W-W neutron star at the center of a huge ($r \approx \infty$), perfectly reflecting cavity. Corresponding to each value of the angular frequency, ω, there are four independent standing-wave modes ($m = -2, -1, 0, +1, +2$). Plotted as a function of ω is the ratio of the total pulsation energy, E_M, of the matter in the star, to the energy, E_W, in one wavelength of the standing gravitational waves. This ratio is the same for all four modes (degeneracy in the quantum number m). (Figure based on numerical computations by Thorne 1968.)

Figure 2 Schematic spacetime diagram for the absorption and reemission of gravitational waves as described by equations (6) - (8). (Based on the work of Price and Thorne 1968.)



E_M / E_W
10^4
10^3
10^2
10
1
10^-1
10^-2
10^-3
10^-4
10^-5
1.5×10^4
2×10^4
3×10^4
4
5
6
7
— ω = 2π × FREQUENCY, sec^-1 →
E_M/E_W × 100
3.5
3
2.5
2
1/τ
20210. 15. 20.
— ω →
E_M/E_W × 100
8
6
4
1/τ
36665. 70. 75.
— ω →
E_M/E_W = (σ/4πτ) / ((ω−σ)^2 + (1/τ)^2)
H-W-W CONFIGURATION
ρ_c = 6 × 10^15 g/cm^3
ℓ = 2 STANDING WAVES



Ib
IIb
IIIb
IV
t=0
IIIa
IIa
STAR
Ia
TIME
RADIUS