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Introduction to Regge and Wheeler "Stability of a Schwarzschild Singularity"

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1. Introduction

The paper "Stability of a Schwarzschild Singularity" by Tullio Regge and John Archibald Wheeler[1] was the pioneering formulation of the theory of weak (linearized) gravitational perturbations of black holes. This paper is remarkable because black holes were extremely poorly understood in 1957, when it was published, and the very name *black hole* would not be introduced into physics (by Wheeler) until eleven years later.

Over the sixty years since the paper's publication, it has been a foundation for an enormous edifice of ideas, insights, and quantitative results, including, for example:

- Proofs that black holes are stable against linear perturbations — both non-spinning black holes (as treated in this paper) and spinning ones.[1–4]
- Demonstrations that all vacuum perturbations of a black hole, except changes of its mass and spin, get radiated away as gravitational waves, leaving behind a quiescent black hole described by the Schwarzschild metric (if non-spinning) or the Kerr metric (if spinning).[5] This is a dynamical, evolutionary version of Wheeler's 1970 aphorism "a black hole has no hair"; it explains how the "hair" gets lost.
- Formulations of the concept of *quasinormal modes* of a black hole with complex eigenfrequencies,[6] and extensive explorations of quasinormal-mode spectra as functions of the black hole's mass and spin.[7]
- Excitations of the quasinormal modes of the final black hole when two black holes collide and merge, and expectations that gravitational waves from those modes will be observed and used to measure the mass and spin of the final black hole and test general relativity's predictions of the black hole's properties.[8]
- Insights into physical structures, made from curved spacetime, that stick out of black holes: so-called *frame-drag vortices* and *tidal tendices* — and their remarkable dynamical evolutions when a black hole is perturbed.[9]

- The discovery of *superradiant scattering* of gravitational waves by a spinning black hole, in which the waves extract spin energy from the hole, so the outgoing, scattered wave is more energetic than the incoming wave.[10,11]
- The discovery, by Zel'dovich, of a corresponding superradiant scattering of gravitational vacuum fluctuations, which manifests itself as spontaneous generation of gravitational waves that feed off the spin energy of a black hole.[10,12]
- Zel'dovich's thereby triggering Stephen Hawking to discover *Hawking radiation* from non spinning as well as spinning black holes.[13]
- The exploitation of quasinormal modes to elucidate the *AdS/CFT correspondence* and issues in quantum gravity.[14]

2. The context in which this paper was written

In 1952 John Wheeler began a transition to relativity from nuclear physics (where, among other things, he had formulated the theory of nuclear fission with Niels Bohr).

His plunge into relativity was triggered, at least in part (see Chap. 10 of his autobiography[15]), by reading the classic 1939 paper by Robert Oppenheimer and Hartland Snyder[16] on the collapse of a highly idealized, spherical, pressure-free star. The spacetime geometry outside the star was the Schwarzschild solution to Einstein's field equations,

$$ds^2 = -(1 - 2m^*/r)dT^2 + (1 - 2m^*/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) = g_{\mu\nu}dx^\mu dx^\nu \ , \qquad (1)$$

where $m^* = 2G\mathcal{M}/c^2$ with $\mathcal{M}$ the star's mass, and G and c Newton's gravitational constant and the speed of light.

As the star's surface shrank, more and more of the Schwarzschild spacetime was opened up to view. Oppenheimer and Snyder showed, analytically, that as measured by a distant observer (for whom T is proper time), the star's surface required infinite time T to reach $r = 2m^*$, but as measured by an observer on the star's surface, it took only finite proper time τ; and they showed analytically that it would be impossible for an external observer to see what happened to the star thereafter. The star would be "cut off" from external view (Oppenheimer's words). Remarkably, Oppenheimer and Snyder did not explore, mathematically, what happened thereafter as seen by observers inside the star or on its surface; but their limited exploration was enough to sow confusion in the minds of even the world's best physicists. As Yevgeny Lifshitz (of Landau and Lifshitz fame) described it to me, many years later, "You cannot appreciate how difficult it was for the human mind to understand how both viewpoints [distant, and stellar-surface] can be true simultaneously."

Not until the early 1960s did the full physics of the collapse's endpoint become clear: the formation of an *event horizon* (also called *future horizon* below) and what Wheeler, in 1968, would call a black hole. Until the early 1960s, confusion reigned.

In their 1957 paper, Regge and Wheeler did not mention the Oppenheimer–Snyder analysis of stellar collapse, but it was likely on their minds to some degree, since that collapse analysis did pique Wheeler's interest in general relativity five years before the Regge–Wheeler paper was submitted, and Wheeler would soon initiate his research project with Kent Harrison and Masami Wakano on the fates of stars with large masses, resulting in a momentous 1958 confrontation between Wheeler and Oppenheimer over the physical outcome of stellar collapse.[17]

In 1957, with collapse and black holes a domain of confusion, Regge and Wheeler chose to focus on a different physical venue for the Schwarzschild metric (1) and its perturbations: A bridge between two asymptotically flat spaces or between two widely separated regions of our own universe, first explored in detail by Albert Einstein and Nathan Rosen in 1935.[18] Wheeler had recently coined the name *wormhole* for this bridge, and that name — like so many of Wheeler's coinages — was quickly embraced by the physics community. In parallel with the Regge–Wheeler research on perturbations of a wormhole, Wheeler and his student Charles Misner were exploring in great detail possible roles of wormholes in physics. In fact, the Misner–Wheeler paper[19] was received at Annals of Physics just four days before the Regge–Wheeler paper,[1] at Physical Review.

In the Regge–Wheeler paper, in addition to *wormhole*, we also see the name *Schwarzschild singularity* used for the object that is perturbed, but we see no explanation of how this name is related to wormholes (or to the Oppenheimer–Snyder analysis of stellar collapse). The Misner–Wheeler paper contains the answer: In their footnote 18, Misner and Wheeler refer to a wormhole as a "bridge across the Schwarzschild singularity", i.e. across the sphere $r = 2m^*$ where the Schwarzschild metric coefficient g_{rr} becomes singular and g_{TT} becomes zero. Although Regge and Wheeler were aware that this was a mere coordinate singularity (after all, a wormhole can extend smoothly through it), they and other relativists of that era were unaware of the past and future horizons that reside at $r = 2m^*$, and unaware of the dynamical evolution that the wormhole must undergo near there. These would get lucidly exposed in 1960 by the work of Wheeler's Princeton colleague Martin Kruskal, in a paper that Wheeler would ghost write for Kruskal,[20] and in related 1958 work of David Finkelstein.[21]

Although the Regge–Wheeler analysis focuses on the stability of a Schwarzschild wormhole, a few years later, when black holes and their horizons were correctly understood, it became clear that the analysis is also applicable, without change, to a Schwarzschild black hole. This is because outside $r = 2m^*$ and at the future horizon $\{r = 2m^*, T = +\infty\}$ the spacetime geometries are identically the same for the black hole and the wormhole.

3. The Regge–Wheeler collaboration

In February 1955, at the 5th Rochester Conferece on High Energy Nuclear Physics, Tullio Regge (then in the second year of his PhD study) was introduced to John Wheeler by Robert Marshak, Regge's PhD advisor and the chair of the University of Rochester Physics Department. Wheeler describes that introduction and subsequent events in his autobiography[15] as follows:

"Marshak described Regge to me as a brilliant mathematical physicist with some interest in general relativity. My conversation with him confirmed Marshak's assessment. It happened, at the time, that I was trying to come to grips with the problem of the stability of what we then called a 'Schwarzschild singularity'. My intuition told me that the Schwarzschild singularity should be stable, but I had not yet been able to prove it. It was important to find out. There is no use looking for such entities in nature if they are not stable. They wouldn't last long enough to be seen. Regge, with his mathematical power, seemed to be just what I needed. I outlined the problem and my thinking about it to him, and he agreed to work with me. ...

"Since I had a vision of how the whole problem should be tackled and how it would work itself out, I sat down and wrote a paper with spaces left for equations, and sent it to Regge. He rose to the occasion, and filled in the blanks. Then, early in 1956, I was able to round up travel funds ... to bring Regge to Leiden [Netherlands, where I was on an 8 month sabbatical, January to September]. We spent ten days together and made the pieces of our work fit smoothly together. Indeed, the Schwarzschild singularity is stable."

Regge, in his short autobiography,[22] remembers the collaboration as beginning in Leiden at the end of his PhD study, which would be in spring 1956, so it appears likely that there was much left to do when Regge arrived in Leiden. Indeed, their paper was not submitted to Physical Review until more than a year later, on July 15, 1957 (though the delay may have been due in large measure to the enormous number of other research projects that Wheeler was pursuing in parallel; see Chap. 12 of Wheeler's autobiography[15]).

4. Overview of the Regge–Wheeler paper

Although the Regge–Wheeler paper was a foundation for the major insights that I list in Sec. 1, and although the mathematical analysis in the paper is correct (aside from typographical errors found by Misner's students Lester Edelstein and C. V. Vishveshwara[23]), and was the template for the analyses in many subsequent papers, nevertheless there are some important conceptual misunderstandings in the Regge–Wheeler paper. These misunderstandings arose from the paper being written in the early era of confusion about the physical nature of the Schwarzschild spacetime.

In this section, I shall walk the reader through the paper, highlighting key, fully correct aspects of the analysis and discussion as well as the conceptual misunderstandings.

4.1. *Section I. Introduction and summary*

In Section I, Regge and Wheeler (R&W) explain that they are focusing on the Schwarzschild spacetime, interpreted physically as a wormhole connecting two regions of our universe that are arbitrarily far apart. They also motivate the idea of a wormhole by describing the closely related Reissner–Nordstrom solution of the Einstein equations, which can be interpreted as a Schwarzschild wormhole with electric field lines threading through it and, via the electrical stresses and energy, modifying the metric. These electrical side remarks give us a sense of why wormholes were of great interest to Wheeler and his students in 1957, and provide a link to the Misner–Wheeler paper[19] that was being submitted for publication almost simultaneously with the Regge–Wheeler paper.[1]

In this Section I, R&W begin refering to the perturbations of the Schwarzschild metric that they will analyze as waves — obviously *gravitational waves*, though they use that full phrase only late in their paper's last section. Although they have waves very much on their minds, they do discover, in their analysis in Section III, that for wavelengths comparable to the size of the Schwarzschild radius $2m^*$ and longer, the perturbations are wavelike only very near $r = 2m^*$ and far from there, $r \gg 2m^*$.

4.2. *Section II. Differential equations in polar coordinates for small first-order changes away from the Schwarzschild metric*

In Section II, R&W develop their mathematical formalism for first-order perturbations $h_{\mu\nu}$ of the Schwarzschild metric. As one might expect, their formalism begins with the separation of variables: writing each perturbed component of the metric as a product of a spherical-harmonic function of angles θ and φ; a monotonic, oscillatory function of time $\exp(-ikT)$; and a to-be-computed function of radius r.

The harmonics that are needed are scalar spherical harmonics for h_{TT}, $h_{Tr} = h_{rT}$ and h_{rr}; vector spherical harmonics on a 2-sphere for $\{h_{T\theta}, h_{T\varphi}\} = \{h_{\theta T}, h_{\varphi T}\}$ and $\{h_{r\theta}, h_{r\varphi}\} = \{h_{\theta r}, h_{\varphi r}\}$; and tensor spherical harmonics on a 2-sphere for $\{h_{\theta\theta}, h_{\theta\varphi} = h_{\varphi\theta}, h_{\varphi\varphi}\}$. R&W build their vector and tensor spherical harmonics in the simplest, most straightforward way possible: by applying 2-sphere gradients once or twice to the scalar spherical harmonics and taking cross products with the Levi–Civita tensor $\epsilon_\mu{}^\nu$ [Eqs. (RW 6)–(RW 11); i.e. Eqs. (6)–(11) in their paper]. The relativity community quickly gave these the name "Regge–Wheeler spherical harmonics", and they have been widely used ever since — though by 1980 a large number of alternative approaches to vector and tensor spherical harmonics were also in use in the relativity community, each with its own powers in special situations.[24]

With their spherical harmonics in hand, R&W focus on perturbations with fixed spherical-harmonic orders L, M and fixed parity — i.e. symmetry + or − under reflection through the origin. They introduce slightly peculiar names for the parity: *even* if the parity is $(-1)^L$ and *odd* if it is $(-1)^{L+1}$. These names were quickly adopted by the relativity community and are widely used even today, though the alternative names *electric-type parity* in place of even and *magnetic-type parity* in place of odd have also gained currency.

R&W write down the most general mathematical form for the metric perturbations of fixed L, M and parity [Eqs. (RW 12) and (RW 13)]. Then, to simplify their subsequent analysis of stability, they specialize to $M = 0$, knowing from standard group theoretic arguments that the eigenfrequencies k of modes must be independent of the harmonic order M. R&W then simplify the mathematical forms of the metric perturbations (RW 12), (RW 13) by means of carefully chosen infinitesimal coordinate transformations (for which they adopt the electromagnetic-like name *gauge transformations* — a name forever thereafter adopted by the relativity community).

For odd parity, they find a gauge (forever thereafter called Regge–Wheeler gauge) in which the perturbed Einstein field equations *decouple* in the following sense: (i) one of the metric perturbations' two unknown radial functions $h_0(r)$ can be expressed in terms of the other $h_1(r)$, and (ii) $h_1(r)$ — or equivalently the quantity $Q(r) = (1 - 2m^*/r)h_1/r$ — satisfies a single ordinary differential equation (RW 24): the radial wave equation, which can be rewritten as

$$d^2Q/dr^{*2} + \left[k^2 - V_{\rm eff}(r)\right] Q = 0\,, \tag{2}$$

where

$$V_{\rm eff}(r) = (1 - 2m^*/r)[L(L+1)/r^2 - 6m^*/r^3] \tag{3}$$

is the effective potential that R&W discuss in Sec. III; and where $r^* = r + 2m^* \ln(r/2m^* - 1)$ is a new radial coordinate that Wheeler later dubs the "tortoise coordinate". From Eq. (2)

(which quickly came to be called the *Regge–Wheeler equation*), it follows immediately that $(\partial^2/\partial r^{*2} - \partial^2/\partial T^2)(Qe^{-ikT}) = V_{\rm eff}(r)$, whence near the Schwarzschild singularity $r = 2m^*$ and far away at $r \gg 2m^*$, where the effective potential is vanishingly small, the perturbations do behave like waves, and propagate along radial rays $r^* = \pm T + \text{constant}$ [cf. Eq. (RW 39)].

Once the Regge–Wheeler equation (2) has been solved, all other aspects of the black hole's perturbations can be computed from the resulting $Q(r)$, or equivalently $h_1(r)$.

For even parity, R&W are not able to decouple the field equations. The best they can do is find a gauge in which there remain three radial functions $H(r)$, $K(r)$ and $H_1(r)$. They can express H_1 in terms of H and K via their Eq. (RW 28), but they then wind up with two coupled ordinary differential equations for H and K. As a result, the even parity analysis becomes much more difficult than the odd parity analysis.

[Remarkably, 12 years later, in 1969, another student of Wheeler's named Frank Zerilli[25] succeeded in decoupling the even parity equations and thereby obtaining a radial wave equation identical to Eq. (2) but with a different effective potential, for a single radial function $Z(r)$, from which all properties of the perturbations could be deduced. Even more remarkably, in 1975 S. Chandraskhar and S. Detweiler[26] discovered a one-to-one mapping between the even parity perturbations and the odd parity perturbations: For even parity there is a single radial function $Q_e(r) = A(k,r)Z(r) + B(k,r)dZ/dr^*$ (for specific functions A and B) that satisfies the Regge–Wheeler equation (2) with the *same* effective potential as for odd parity. From any solution of this equation, one can construct all the properties of the corresponding even parity perturbations as well as of the odd parity perturbations, and because the effective potential is the same as for odd parity, the spectrum of frequencies of the even parity modes is identical to that of the odd parity modes! The mathematical route to this amazing result was so complex that it is no wonder R&W, in their pioneering foray into this problem, failed to find it.]

With the even parity and odd parity radial equations in hand, R&W focus on static (time-independent) perturbations. They easily show, as one might expect, that for the even parity perturbations the static, spherically symmetric $M = 0$ perturbation represents a simple change of mass of the wormhole; and the static, dipolar $M = 1$ perturbation represents a simple physical displacement of the wormhole from its original position. For the odd parity perturbation, they miss the fact that the static, dipolar $M = 1$ perturbation represents adding a small spin angular momentum to the wormhole.[2]

For $L \geq 2$, R&W note that one class of static perturbations represents deformations of the wormhole by the gravity of external masses, but they miss a second class of static perturbations, which represent *singular* multipolar pertiurbations of the Schwarzschild singularity $r = 2m^*$ (the wormhole's throat and horizon). Again, missing this is not at all surprising, considering relativists' confusion about the physical nature of $r = 2m^*$ in 1957.

4.3. *Section III. Boundary conditions and stability*

In Sec. III, R&W focus on dynamical perturbations of a wormhole. Dynamical perturbations are possible only for spherical harmonics of quadrupole order or higher, $L \geq 2$.

R&W require, of course, that at some initial moment of time the perturbations be regular at both $r = 2m^*$ and $r = \infty$, but they do not discuss what "regularity" means, physically. Instead, they examine the solutions to their odd parity radial wave equation (2) and see that for real frequencies k the solutions $Q(r^*)$ are all oscillatory with asymptotically finite amplitude at both $r = 2m^*$ ($r^* = -\infty$) and $r = +\infty$; and they assume (correctly it turns out) that these solutions are physically regular. A proof of physical regularity would entail, for example, examining the components of the Riemann curvature tensor as measured by observers far from the Schwarzschild singularity and by observers falling into the Schwarzschild singularity (falling through the future horizon).

Continuing to focus on odd parity and real frequencies k, R&W discuss three types of solutions of their radial wave equation:

- Case 1: High frequency solutions, $k \gtrsim 1/m^*$, which are oscillatory at all radii. These represent waves that propagate freely, little affected by the effective potential (by the wormhole's spacetime curvature) from $r = 2m^*$ to $r = \infty$ and the reverse. Today we recognize them as waves that propate nearly freely up from the wormhole's past horizon and on outward to $r = \infty$, or from $r = \infty$ inward to and down the wormhole's future horizon.
- Case 2a: Low frequecy solutions, $k \ll 1/m^*$, which have large oscillatory amplitudes near $r = 2m^*$ ($r^* \to -\infty$), die out inside the effective potential, and have tiny oscillatory amplitudes at $r \to \infty$. R&W identify these as waves that are trapped near the Schwarzschild singularity much like the electromagnetic waves trapped in *geons* that Wheeler had recently conceived and explored.[27] Today we recognize them as waves that come up from the wormhole's past horizon, and reflect off the Schwarzschild spacetime curvature where the effective potential is large, so almost all of the wave energy goes down into the wormhole's future horizon but a tiny portion leaks out and travels toward $r = \infty$; or the time reversal of this.
- Case 2b: Low frequency solutions, $k \ll 1/m^*$, which have large oscillatory amplitudes at large radii $r \gg 2m^*$, die out inside the effective potential, and have tiny oscillatory amplitudes near $r = 2m^*$ ($r^* \to -\infty$). Today we recognize these as waves that impinge on the wormhole from $r = \infty$ and are largely reflected by the wormhole's spacetime curvature, with a tiny amount of transmission toward and down the future horizon; or the time reversal of this.

R&W then discuss waves that "go through the wormhole", and infer, correctly of course, that the phases of these waves are arbitrary. Throughout this discussion, and throughout their paper, R&W assume that the wormhole is static. But it is not. As Wheeler himself would explain with great clarity in 1960, in a paper he would ghostwrite for Martin Kruskal,[20] the Schwarzschild wormhole is dynamical. The waves that R&W think of as going into and through the wormhole actually go into and down the future horizon and get caught and destroyed in the wormhole's interior, future singularity. And those that R&W think of as coming to us from the wormhole's other mouth actually originate in the wormhole's past singularity, and travel up through the wormhole's past horizon and onward toward $r = \infty$ (with some reflection off the spacetime curvature).

In their last subsection, R&W address the problem of the wormhole's stability.

Focusing again on the odd parity case, they easily show that for imaginary frequencies k

all solutions $Q(r^*)$ of their odd parity wave equation grow without bound either as $r \to 2m^*$ (the wormhole throat) or as $r \to \infty$, and so are unphysical. For the even parity case, they state without proof that a detailed but more complex analysis shows that all solutions with imaginary frequency are unphysical. Based in these results, they assert that a wormhole (or Schwarzschild singularity) is stable against weak perturbations.

What R&W do not say but could have, to make this more firm, is that their odd parity radial wave equation (2) with boundary conditions of regularity on $Q(r^*)$ at $r = 2m^*$ and $r = \infty$ is a self-adjoint eigenvalue problem for k^2; and by self adjointness k^2 must be real, so any instability must show up as an imaginary frequency k with regular $Q(r^*)$. The self adjointness further implies that the modes with real frequency form a complete set, in terms of which any regular initial perturbation can be expanded and evolved; and their stability implies stability of evolution of that regular initial perturbation.

Even after taking account of self adjointness, there are several holes in the R&W stability proof — holes that were filled in by others over the ensuing years and decades:

- R&W did not prove that *physical* regularity at $r = 2m^*$ and at $r = \infty$ implies regularity of $Q(r^*)$ there. In 1970, with the past and future horizons well understood, Vishveshwara fixed this hole.[2]
- R&W did not succeed in constructing a self-adjoint eigenvalue problem for k^2 in the even parity case. In 1970 Zerilli fixed this hole.[25]
- The problem of stability for a black hole that is formed by gravitational collapse of a star is actually very slightly different from that for the pure vacuum Schwarzschild spacetime, which is truly a wormhole: the past horizon is replaced by the world tube of the star that collapsed long long ago. For ordinary theoretical physicists like me, that difference seems exceedingly unlikely to be important, and I glossed over it above; but in 2015 Kay and Wald gave a rigorous proof of stability for such a black hole, taking that difference into account.[28]

This level of rigor is enough to satisfy almost all physicists. However mathematicians and the most mathematical of physicists are focusing today on attempts to prove stability of a Schwarzschild wormhole in stronger senses, what they call "linear and nonlinear stability". For a detailed discussion and progress report, see a recent monumental paper by Klainerman and Szeftel.[29]

5. Quasinormal modes and their stability

In 1970 Misner's student Vishveshwara simulated[30] the scattering of wave packets of gravitational waves off a Schwarzschild black hole by solving numerically the odd parity Regge–Wheeler radial wave equation (2). For wave packets with widths comparable to the black hole's size, the scattered waves oscillated sinusoidally with rapidly decaying amplitude. The following year my student William Press performed other numerical experiments showing similar results and interpreted the scattered waves as coming from what he called *quasinormal modes* of vibration of the black hole, which were excited by the incoming waves.[6]

These quasinormal modes have boundary conditions of no upcoming waves at $r = 2m^*$

and no incoming waves at $r = \infty$. The waves are purely downgoing at the horizons and outgoing far from the black hole.

Any quasinormal mode can be constructed from a superposition of modes with real frequencies k, and so is guaranteed to be stable.

In 1975 Chandrasekhar and Detweiler[26] carried out the first numerical computations of the complex eigenfrequencies of a Schwarzschild black hole's (or wormhole's) quasinormal modes. For subsequent history, in extreme brevity, see Sec. I of this paper; for greater detail see the 2015 review article by Teukolsky[4] and references therein.

6. A personal conclusion

(Kerr)

I conclude with some personal remarks. As a graduate student of John Wheeler in 1962–65, and then when building my own research group at Caltech in the late 1960s and early 1970s, I was greatly influenced by the Regge–Wheeler paper. I pushed my students to generalize the R&W analysis to spinning black holes, and they did so: Teukolsky separated variables and decoubled the resulting perturbation equations to obtain a radial wave equation for a variable $R(r)$ from which all properties of the perturbations could be deduced;[31] and Press and Teukolsky proved (with a mixture of numerical and analytical manipulations of the Teukolsky equation) that a Kerr black hole is stable.[3] I myself, with colleague Alfonso Campolattaro and student Richard Price, generalized the R&W analysis to study perturbations and quasinormal modes of nonrotating relativistic stars.[32,33] I helped my friend and colleague James Hartle extend the R&W analysis of time-independent perturbations to stars, which we set into rotation and explored to second order in their rotational angular velocity.[34,35] And a later generation of my students, beginning with Fernando Echeverria,[36] focused on the gravitational waves from black-hole normal modes as targets for gravitational wave detectors.

I therefore owe a huge debt of personal gratitude to Tullio Regge and John Archibald Wheeler for their pioneering work on perturbations of the Schwarzschild spacetime.

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