10

The Gravitational Waves That Bathe the Earth: Upper Limits Based on Theorists' Cherished Beliefs†‡

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Abstract

On the basis of our cherished beliefs about the structure of the Universe and the theory of gravitation, we derive theoretical upper limits on the strengths of the gravitational waves that bathe the Earth. Separate limits are presented, as functions of frequency, for waves from extragalactic sources and for waves from inside our own galaxy; and in each case, for discrete sources (bursters, transient sources, and monochromatic sources) and for a stochastic background due to unresolved sources. An observation of gravitational waves exceeding these limits would be disturbing (and exciting), since it would require a modification of one or more generally accepted assumptions about the astrophysical universe or the nature of gravity.

I. Introduction

During the past two decades general relativity theory has had an increasingly strong impact on astrophysics—first in the theory of quasars;

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then in cosmology, pulsars, compact x-ray sources, and the search for black holes. We hope for an even stronger impact in the future, when gravitational waves open up a new "window" onto the Universe—a window in which general relativity will play an absolutely essential role.

The efforts of experimenters to develop gravitational-wave detectors of ever-increasing sensitivity have been described in a number of recent review articles [1–4]. As these efforts proceed, it is useful to have theoretical "benchmarks" against which to gauge their progress. Such benchmarks are of three major types. The first type, as sensitivities improve, are "nihil obstat" upper limits on the strengths of the waves. An observation of waves above these limits would overturn one or more cherished beliefs about either the structure of the Universe or the physical laws governing gravitational radiation. Type-two benchmarks are at a level where the best estimates of plausible astrophysical sources indicate that something should be seen. Observations at these sensitivities are sure to give significant astronomical information; even if no waves are detected, many otherwise acceptable models will be eliminated. Type-three benchmarks are the absolute minimum gravitational-wave strengths consistent with other astronomical observations. A failure to see waves below these limits would be as serious a matter as observations of waves above the type-one limits; in either case, something is radically wrong with the theory of gravitation or with conventional astrophysical wisdom.

Type-two and type-three benchmarks have been reviewed in several recent articles [5,6]. The purpose of this article is to set forth benchmarks of the first type—"cherished-belief" upper limits on gravitational-wave strengths.

In Section II we list and discuss the cherished beliefs on which our limits are based. In Section III we derive, from those cherished beliefs, upper limits on the strength of any stochastic background of gravitational waves that might bathe the Earth—both a limit on waves from unresolved sources in our galaxy, and a limit on extragalactic waves. We also describe scenarios that could lead to these upper limits. In Section IV we derive similar upper limits on waves from discrete sources including bursters, transient sources, and monochromatic sources. Again there are separate upper limits for sources in our own galaxy and extragalactic sources. For the case of broad-band bursts, we also describe a scenario that could lead to the galactic upper limits.

Throughout we shall restrict attention to gravitational-wave frequencies in the domain of current experimental interest: 10^{-4} Hz $\lesssim f \lesssim 10^{+4}$ Hz. The lower limit 10^{-4} Hz is dictated by the technology of gravitational-wave detectors [1–4]—in particular, the round-trip radio-wave travel time to spacecraft at reasonable distances (e.g., Jupiter). The upper limit 10^{+4} Hz is dictated by our *cherished belief* [5] that the only highly efficient sources of gravitational waves in the Universe today are objects near their Schwarzschild radii—neutron stars and black holes of stellar mass and larger—and that these objects cannot radiate significantly at frequencies above $f_{max} \simeq 10^{+4}$ Hz.

The notation used in our discussion is summarized in the Appendix. A more detailed discussion of each parameter is given at the point in the text where it is first introduced.

II. Cherished Beliefs

The cherished beliefs, on which we base our limits, are of two types: beliefs about the astrophysical structure of the Universe (Section II.A), and beliefs about the physical laws governing gravitational radiation (Section II.B).

A. The Structure of the Universe

Our first cherished belief is the *cosmological principle* that *we do not live in a special time or place in the Universe*—except for being inside a local density enhancement, the galaxy. The cosmological principle implies that, on the average, sources of gravitational waves are no more luminous now than they have been (and will be) for a Hubble time $T_H = 1 \times 10^{10}$ years, the only time scale available. It also implies that the nearest source is at a typical distance from us, neither fortuitously near nor far. (For objects of number density n in Euclidean three-space, the mean distance to the nearest one is $0.55396 \ldots n^{-1/3}$; over 90% of the time, the nearest is between $0.2n^{-1/3}$ and $0.9n^{-1/3}$. We shall use $0.5n^{-1/3}$ as the distance to the nearest source throughout this paper.)

Our next cherished belief is that there is *no significant amount of "relict," primordial gravitational radiation* bathing the Earth—more precisely, that all the significant sources of gravitational waves are at cosmological redshifts $z \lesssim 3$. This is as much a simplifying assumption as a cherished belief: Although semiplausible models of the early Universe give only modest amounts of gravitational radiation [7,8] (amounts well below the upper limits of this paper), we are so ignorant about the early Universe that it is hard to place *firm* upper limits on the waves from there, except the obvious limit that their total energy density not exceed by much the density required to close the Universe. The closure limit will follow from our other cherished beliefs without explicitly assuming it.

The cosmological principle, plus the belief in "no primordial waves," allows us to approximate the Universe by a very simple model that is accurate to within an order of magnitude in energies (a factor of three in gravitational-wave amplitudes). In this model the expansion of the Universe is ignored, space is regarded as Euclidean, the Universe is regarded as extending outward from Earth in all directions to a Hubble distance $R_H = cT_H = 1 \times 10^{10}$ light years, within this distance the smeared-out mass density of potential gravitational-wave sources is regarded as constant and as equal to the "closure density" $\rho_u = (c^2/G)(3/8\pi)R_H^{-2} = 2 \times 10^{-29}$ g/cm^3, and outside R_H the density drops to zero (cosmological cutoff on sources). Our use of the closure density for ρ_u does not mean that we believe in this value, but rather that this is a reasonable upper limit and will thus give rise to the largest possible limits on gravitational-wave strengths. The galaxy we shall model as a region of constant, enhanced mean mass density, $\rho_g = 2 \times 10^{-24}$ g/cm^3 (no radial structure), and of spherical shape with radius $R_g = 60{,}000$ light years and with the Earth located (roughly) at its center. The numbers for our galaxy take account of a now popular galactic halo with total mass $M_g = (4\pi/3)R_g^3\rho_g \simeq 1 \times 10^{12}\ M_\odot$ and radius $R_g \simeq 60{,}000$ light years [9–11].

Our third cherished belief is that *within our galaxy no single, coherently radiating object has mass in excess of* $M_{max} \simeq 10^8\ M_\odot$ [12]. This is a very generous upper limit. We make no assumption about the maximum mass of extragalactic objects.

Our fourth cherished belief is that the dominant sources of gravitational waves have *no significant beaming* of their radiation. In principle, strong beaming can occur—e.g., in waves from ultrarelativistic collisions of astrophysical objects [13,14], in waves from sources with gravitational lens properties [15,16], and in waves from carefully contrived directional antennas [17]. However, we do not know of any type of hypothetical strong-beaming source that is likely to make up a significant fraction of the mass density of the galaxy or Universe. Moreover, our limits are fairly insensitive to the no-beaming assumption: A simple geometrical analysis in flat space shows that, if sources beam their energy into a solid angle $\Omega < 4\pi$, and if the Earth is located randomly relative to the beams, then the expected energy flux from the nearest visible object increases only as $(\Omega/4\pi)^{-1/3}$, and the expected total flux from all sources out to some fixed cutoff radius remains constant. (On the other hand, as Lawrence [15], Misner [18], Jackson [16], and others have argued, there *could* be an object at the center of our galaxy that preferentially beams its radiation into the galactic plane where we lie. Our no-beaming assumption rules this out.)

Our fifth cherished belief is that narrow band sources of gravitational waves($\Delta f \ll f$) have their *frequencies f distributed randomly over a bandwidth $\Delta f \gtrsim f$.*

B. *The Physical Laws Governing Gravitational Radiation*

We take our cherished beliefs about gravitational-wave theory from general relativity, though most other relativistic theories of gravity would lead to similar beliefs. Our beliefs are expressed in order-of-magnitude form.

Consider a source of mass M, which radiates gravitational waves coherently. (Examples: A pulsating star, a binary star system, two colliding black holes.) If small parts of the source produce waves that superpose incoherently, those parts must be regarded as separate sources. (Example: for the thermal bremsstrahlung radiation produced by collisions of electrons and ions inside the sun, the source is not the entire sun but rather a single colliding electron–ion pair.) Let f be a frequency at which the source radiates significantly. Our first cherished belief is an *upper limit on the frequency f, for a given source mass M* [5]:

$$f \lesssim \frac{c^3/G}{2\pi M} \simeq \frac{30{,}000\ \text{Hz}}{M/M_\odot}. \tag{1}$$

This limit corresponds to a belief that the characteristic time scale $(2\pi f)^{-1}$ of the coherent waves must exceed the light-travel time across half the Schwarzschild radius of the source GM/c^3. This limit can be violated in sources with significant beaming—e.g., sources with ultrarelativistic internal velocities [13,14]; but we have ruled out such sources. We strongly doubt that coherent, nonbeaming sources can violate this limit. For example, typical events involving black holes (births, collisions, infall of matter) produce waves of frequency $f \sim 10{,}000\ \text{Hz}\ (M/M_\odot)^{-1}$ [5,19–21], with a very rapid falloff of intensity above $f = 30{,}000\ \text{Hz}\ (M/M_\odot)^{-1}$.

In general relativity and other similar theories, a source with negligible beaming gives rise predominantly to quadrupole radiation. The luminosity of such a source is given by Einstein's [22] quadrupole formula

$$L = \frac{1}{5}\left(\frac{G}{c^5}\right)\left(\sum_{j,k}\frac{\partial^3 I\!\!\!-_{jk}}{\partial t^3}\right)^2,$$

where the third time derivative of the quadrupole moment, expressed in terms of the coherent source's mass M, radius R, and frequency f, is

$$\frac{\partial^3 I\!\!\!-_{jk}}{\partial t^3} \lesssim MR^2(2\pi f)^3 \lesssim 2\pi M f c^2.$$

Here we have used the relation, for a coherent source,

$$(2\pi f R) \simeq (\text{internal velocity of source}) \lesssim c.$$

Combining these relations we obtain a cherished belief about the *maximum luminosity that a source of mass M and frequency f can produce*

$$L \lesssim \frac{G}{c}(2\pi M f)^2 \simeq (4 \times 10^{50}\ \text{erg/sec})\left(\frac{M}{M_\odot}\right)^2\left(\frac{f}{1\ \text{Hz}}\right)^2. \tag{2}$$

Note that when $f = f_{\max} = (c^3/G)(2\pi M)^{-1}$, then $L \lesssim L_{\max} \simeq c^5/G \approx (4 \times 10^{59}\ \text{erg/sec})$—a limit which, so far as we know, was first suggested by Dyson [23].

In our analysis we shall idealize our typical source as radiating gravitational waves in a series of outbursts separated by quiescent periods. Let N denote the total number of outbursts, τ_* the mean duration of each outburst, and L the average luminosity during each outburst. Our next cherished belief is that, in the source's entire lifetime, *the total energy radiated cannot exceed the total mass-energy Mc^2 of the source*

$$NL\tau_* \leq Mc^2. \tag{3}$$

In describing the gravitational waves arriving at Earth we shall use, at various times, four different measures of wave strength: *First*, in describing waves from discrete sources we shall use a mean value h for the dimensionless gravitational-wave amplitude at the frequency f in a bandwidth $\Delta f \simeq f$:

$$h \simeq \langle [h_+(t)]^2 + [h_\times(t)]^2 \rangle^{1/2}.$$

Here the average $\langle\ \rangle$ is over the time τ_* that the source is on; and $h_+(t)$ and $h_\times(t)$ are the dimensionless amplitudes for the two orthogonal modes of polarization, which for a source in the z direction determine the transverse-traceless part of the metric perturbation via

$$h^{TT} = h_+(t - z)[\mathbf{e}_x \otimes \mathbf{e}_x - \mathbf{e}_y \otimes \mathbf{e}_y] + h_\times(t - z)[\mathbf{e}_x \otimes \mathbf{e}_y + \mathbf{e}_y \otimes \mathbf{e}_x].$$

We presume that h_+ and $h_\times$ have been sent through a bandpass filter of frequency f and bandwidth $\Delta f \simeq f$. For monochromatic waves, $h_+(t) = A_+ \cos(2\pi f t + \phi_+)$ and $h_\times(t) = A_\times \cos(2\pi f t + \phi_\times)$, our definition of h gives $h = [\frac{1}{2}(A_+^2 + A_\times^2)]^{1/2}$. *Second*, for discrete sources we shall also use the total flux of energy $\mathcal{F}$ at the frequency f and in the bandwidth $\Delta f \simeq f$. We shall assume (cherished belief!) the general relativistic *relationship between $\mathcal{F}$ and h*:

$$\mathcal{F} = \frac{c^3}{G}\frac{\pi}{4} f^2 h^2 = \left(0.03\ \frac{\text{erg}}{\text{cm}^2\ \text{sec}}\right)\left(\frac{f}{1\ \text{Hz}}\right)^2\left(\frac{h}{10^{-20}}\right)^2. \tag{4}$$

Third, in describing stochastic background radiation, we shall use the energy flux per unit frequency $\mathcal{F}_f$ (flux density; erg/cm^2 sec Hz), which our cherished beliefs imply will be independent of the orientation of our unit surface area. *Fourth*, for the stochastic background we shall also use an amplitude $\tilde{h}(f)$ (dimensions $\mathrm{Hz}^{-1/2}$), which is defined in analogy with Eq. (4) by

$$\mathscr{F}_f = \frac{c^3}{G}\frac{\pi}{4} f^2 \tilde{h}^2 = \left(0.03 \frac{\mathrm{erg}}{\mathrm{cm^2\,sec\,Hz}}\right)\left(\frac{f}{1\ \mathrm{Hz}}\right)^2\left(\frac{\tilde{h}}{10^{-20}\ \mathrm{Hz}^{-1/2}}\right)^2. \quad (5)$$

The square of $\tilde{h}$, roughly speaking, is the spectral density of the gravitational-wave amplitude $h(t)$. The stochastic background will produce in a broadband gravitational-wave detector a spectral density of strain $(\Delta l/l)_f^2 = \alpha \tilde{h}^2$, where α is a factor of order unity that depends on the detailed construction of the detector.

In relating the strengths of the waves at earth to the luminosity L of a source at distance r, we shall assume *energy conservation* (cherished belief!)

$$\mathscr{F}_{\text{due to one source}} = \frac{L}{4\pi r^2}, \quad (6)$$

and we shall assume that *gravitational waves propagate at the speed of light* (cherished belief!).

III. Upper Limits on Stochastic Background

From the cherished beliefs of Section II, one can derive the upper limits on a stochastic background of gravitational radiation shown in Fig. 1. In Section III.A we explain the origin of the limit for extragalactic radiation; in Section III.B we explain the galactic limit.

A. Extragalactic Radiation

Consider a specific frequency f at which the background is strong, and let Δf be the bandwidth about f over which the specific flux $\mathscr{F}_f$ is roughly constant. For a background due to broadband sources, by definition of "broadband," we have $\Delta f \gtrsim f$. For a background due to superposed narrowband sources, the last cherished belief of Section II.A ("frequencies distributed randomly over a bandwidth $\gtrsim f$") implies $\Delta f \gtrsim f$. Thus, in either case the background is roughly constant over $\Delta f \gtrsim f$, but it can drop off fairly rapidly at both ends of this band.

An upper limit on $\mathscr{F}_f$, for extragalactic background, follows from our cherished beliefs that (i) the total energy radiated by all sources cannot exceed the sum of the masses of those sources; and (ii) we do not live at a special place or time, so that the total gravitational-wave energy must be spread roughly uniformly over the entire Universe and the energy density at

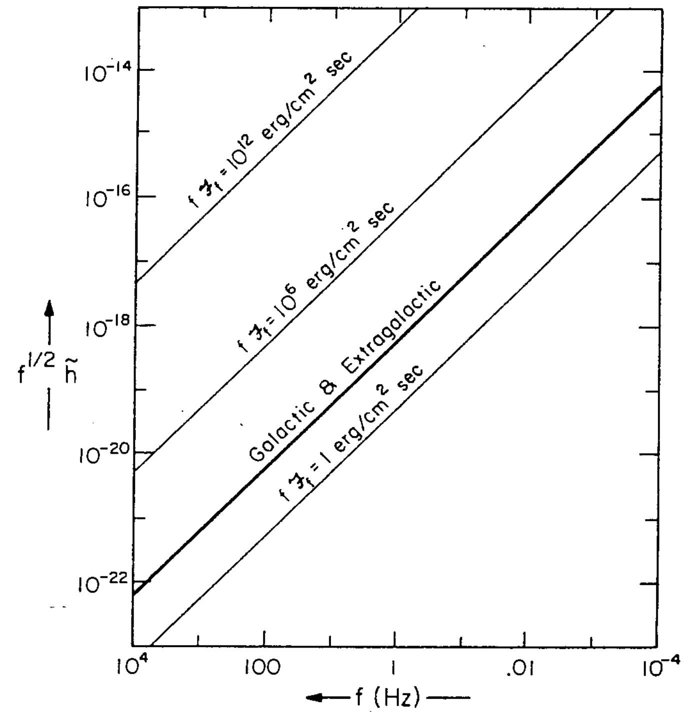

Fig. 1 Upper limits on a stochastic background of gravitational radiation at Earth [Eqs. (7) and (8)]. The limit for radiation from sources in our galaxy is approximately the same as the limit for extragalactic radiation. For notation, see Eq. (5) of text and associated discussion.

Earth must be roughly the same as the average energy density in the Universe. These beliefs imply a total energy density in background radiation at Earth less than or of order the total mass-energy density of the Universe:

$$\begin{pmatrix}\text{background}\\ \text{energy density}\end{pmatrix} = \frac{4\mathcal{F}_f \Delta f}{c} \lesssim \rho_u c^2.$$

(The factor 4 comes from integrating over all directions.) Combining this with the bandwidth requirement $\Delta f \gtrsim f$, we obtain the limit

$$\mathcal{F}_f \lesssim \frac{1}{4}\left(\frac{\rho_u c^3}{f}\right) \simeq \left(100 \frac{\text{erg}}{\text{cm}^2\,\text{sec Hz}}\right)\left(\frac{f}{1\text{ Hz}}\right)^{-1}, \tag{7a}$$

which corresponds to a wave amplitude

$$f^{1/2}\tilde{h} \lesssim 6 \times 10^{-19}(f/1\text{ Hz})^{-1} \tag{7b}$$

[cf. Eq. (5)]. These limits are shown in Fig. 1.

These extragalactic limits are widely accepted and often discussed in the astrophysical literature (see, e.g. [24]).

The upper limit (7) can be achieved, within the framework of our cherished beliefs, in a variety of ways. For example, at any frequency $f \lesssim 10{,}000$ Hz the following scenario is allowed, though not likely: Early in the evolution of the Universe a sizable fraction of the Universe's mass might have gone into black-hole binary systems of mass $M \sim (c^3/G)(2\pi f)^{-1}$. Under the action of gravitational radiation reaction, the holes in each binary will spiral together, releasing a sizable fraction of their mass M in a final burst of broadband radiation of frequency f and duration $\tau_* \sim (2\pi f)^{-1}$. These bursts must be randomly distributed over the volume of the Universe and over the Hubble time, so that the average number of bursts occurring at any given time is

$$\mathcal{N} = \frac{(4\pi/3)\rho_u R_H^3}{M} \frac{\tau_*}{T_H} \sim 1$$

(cf. relations in Section A of the Appendix). This is also the average number of bursts passing Earth at each moment of time; and these bursts give rise to background radiation near the upper limit (7).

One can also achieve these upper limits by a superposition of many bursts with lower individual intensities and longer individual durations.

B. *Galactic Radiation*

For galactic background radiation, as for extragalactic, the bandwidth over which $\mathcal{F}_f$ is large must be $\Delta f \gtrsim f$. The radiation must be spread roughly uniformly over the interior of the galaxy ("no special place"), so that the total radiation energy in the galaxy is $(4\mathcal{F}_f\, \Delta f)(4\pi/3)(R_g^3/c)$. This radiation energy will escape from the galaxy in a time R_g/c and must be replenished by source emission in that time. The total energy emitted during the Hubble time T_H is thus $(cT_H/R_g) \times$ (total energy now in galaxy); and this cannot exceed the total mass-energy of the galaxy $(4\pi/3)R_g^3\rho_g c^2$. Combining these constraints we obtain the upper limit

$$\mathcal{F}_f \lesssim \frac{1}{4}\frac{R_g \rho_g c^2}{f T_H} \simeq 100 \frac{\text{erg}}{\text{cm}^2 \text{ sec Hz}} \left(\frac{f}{1 \text{ Hz}}\right)^{-1}, \tag{8a}$$

which corresponds to a wave amplitude

$$f^{1/2}\tilde{h} \lesssim 6 \times 10^{-19}(f/1 \text{ Hz})^{-1}. \tag{8b}$$

Note that this is the same order-of-magnitude limit as we obtained for extragalactic radiation! It is the same by virtue of the coincidence (or is it a coincidence?) that the closure density ρ_u and Hubble distance $R_H = cT_H$ of the

Universe, and the density ρ_g and radius R_g of the galaxy satisfy

$$\rho_u \sim \rho_g\left(\frac{R_g}{R_H}\right).$$

The upper limit (8) can be achieved, within the framework of our cherished beliefs, by putting the bulk of the mass of our galaxy into objects of mass $\sim M$ [with $M \lesssim (c^3/G)(2\pi f)^{-1}$ and $M \lesssim M_{max} = 10^8\, M_\odot$], which radiate away all their mass in bursts of mean frequency f, duration τ_*, and luminosity $L = Mc^2/\tau_*$ [with $L \lesssim (G/c)(2\pi M f)^2$ and $\tau_* \lesssim T_H$]. The locations of these objects and the epoch of their emission must be randomly distributed through the galaxy; so the number of "on" sources contributing to $\mathscr{F}_f$ at Earth at any given time will be $\mathscr{N} \simeq (M_g/M)(\tau_*/T_H)$ [where $\mathscr{N} \gtrsim 1$ so that experimenters will see a background rather than individual events]. The mass M and burst duration τ_* can be chosen in accord with our cherished-belief constraints (items in brackets above) so long as

$$f \gtrsim f_{min} \simeq \left(\frac{c^3/G}{4\pi^2 M_{max} T_H}\right)^{1/2} \simeq 1 \times 10^{-11}\ \text{Hz}; \tag{9}$$

and thus for these frequencies our cherished beliefs cannot give any limit tighter than (8). Note that $f \gtrsim f_{min}$ includes all frequencies of experimental interest. At lower frequencies, objects of mass $M \lesssim M_{max} = 10^8\, M_\odot$ radiating with luminosities $L \lesssim (G/c)(2\pi M f)^2$ cannot radiate away all their mass-energy Mc^2 in a time τ_* less than the age of the Universe T_H; and, consequently, the maximum galactic flux density and wave amplitude are reduced from the limit (8) to

$$\mathscr{F}_f \lesssim \frac{1}{4}\frac{R_g \rho_g c^2}{f T_H}\left(\frac{f}{f_{min}}\right)^2 \simeq 10^{13}\,\frac{\text{erg}}{\text{cm}^2\ \text{sec Hz}}\left(\frac{f}{f_{min}}\right), \tag{10a}$$

$$f^{1/2}\tilde{h} \lesssim 6 \times 10^{-19}(f/1\ \text{Hz})^{-1}(f/f_{min}) \simeq 6 \times 10^{-8}, \tag{10b}$$

for

$$f \lesssim f_{min} \simeq 1 \times 10^{-11}\ \text{Hz}.$$

However, this range of frequencies is outside the domain of interest for the present discussion (10^{-4} Hz $\lesssim f \lesssim 10^{+4}$ Hz).

IV. Upper Limits on Waves from Discrete Sources

We turn now to gravitational waves from discrete (resolved) sources, including broadband bursts (duration $\tau_* \sim 1/2\pi f$); transient sources ($1/2\pi f \ll \tau_* < \hat{\tau}$, where $\hat{\tau}$ is the total observation time, i.e., the total time that the experimenter searches for gravitational waves); and permanent sources

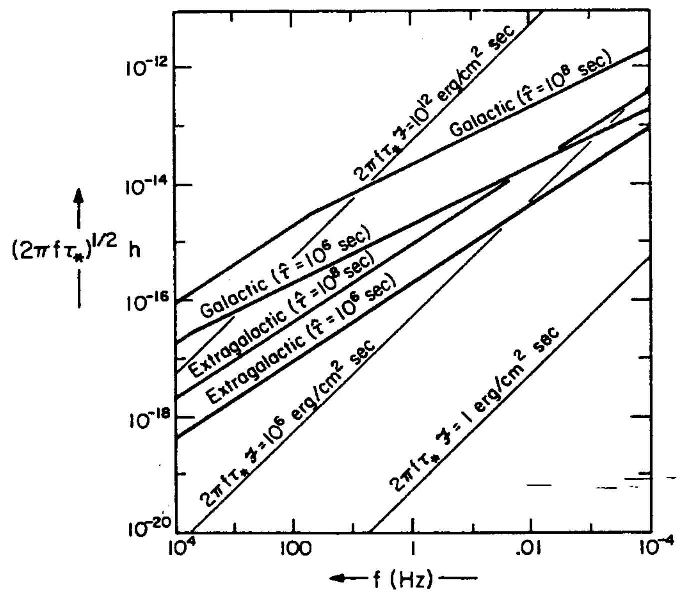

Fig. 2 Upper limits on discrete sources of gravitational waves [Eqs. (11)–(14). These limits answer the following question: "An Experimenter searches, with total observation time τ, for a discrete gravitational-wave event of duration $\tau_* \leq \tau$ at frequencies $f > 1/\tau_* \geq 1/\hat{\tau}$ in a bandwidth $\Delta f \simeq f$. What is the flux $\mathcal{F}$ and amplitude h [Eq. (4) and associated discussion] of the strongest single event he can hope to see, within the constraints of our cherished beliefs?"

($\tau_* \gtrsim \hat{\tau}$). The transient and permanent sources can be either broadband ($\Delta f \sim f$) or narrowband ($\Delta f \ll f$). Our characterization of the waves by their flux $\mathcal{F}$ and amplitude h pays no attention to the bandwidth of the source. Since the experimenter can never know the total "on time" τ_* of the source unless $\tau_* < \hat{\tau}$, and since our cherished beliefs allow stronger waves the shorter is τ_*, we can restrict attention to the case $\tau_* \leq \hat{\tau}$.

For discrete sources our upper limits answer the following question: "An experimenter searches, with total observation $\hat{\tau}$, for a gravitational-wave event of duration $\tau_* \leq \hat{\tau}$ at frequencies $f > (2\pi\tau_*)^{-1}$ in a bandwidth $\Delta f \simeq f$. What is the flux $\mathcal{F}$ and amplitude h of the strongest single event he can hope to see within the constraints of our cherished beliefs?" The upper limits that answer this question are shown in Fig. 2. These limits are derived and discussed, for extragalactic sources, in Section IV.A, and for galactic sources in Section IV.B.

A. Extragalactic Sources

Let the frequency f, event duration τ_*, and observation time $\hat{\tau}$ be given. The waves will be strongest if the bulk of the mass of the Universe resides in sources of some optimally chosen mass M [with $M \lesssim (c^3/G)(2\pi f)^{-1}$], each of which produces some optimal luminosity L during its "on time" τ_* [where $L \lesssim (G/c)(2\pi M f)^2$], and each of which has some optimal number N of "on events" during the Hubble time T_H [with $NL\tau_* \leq Mc^2$].

The number density of sources is $n \simeq \rho_u/M$, and the probability that a given source will turn on during the observation time $\hat{\tau}$ is $P = N\hat{\tau}/T_H$. Consequently, the nearest source that turns on during $\hat{\tau}$ is at a distance

$$r \simeq \frac{1}{2}(nP)^{-1/3} = \frac{1}{2}\left(\frac{MT_H}{\rho_u N\hat{\tau}}\right)^{1/3};$$

and the flux produced at Earth by this nearest (and thus strongest) source is

$$\mathscr{F} = \frac{L}{4\pi r^2} = \frac{1}{\pi}\left(\frac{N^2L^3}{M^2}\right)^{1/3}\left(\frac{\rho_u\hat{\tau}}{T_H}\right)^{2/3}.$$

This flux is maximized, subject to our cherished-belief constraints (brackets above) by setting $N = 1$, $M \simeq (c^3/G)(2\pi f)^{-1}$, and $L \simeq Mc^2/\tau_*$ [corresponding to $L \simeq (2\pi f\tau_*)^{-1}(G/c)(2\pi Mf)^2 \lesssim (G/c)(2\pi Mf)^2$]. The resulting upper limit is

$$\begin{aligned}\mathscr{F} &\lesssim \left(\frac{1}{2\pi f\tau_*}\right)\left(\frac{c^3}{\pi G^{1/3}}\right)\left(\frac{\rho_u 2\pi f\hat{\tau}}{T_H}\right)^{2/3} \\ &\simeq \left(\frac{1\times 10^7\ \mathrm{erg/cm^2\ sec}}{2\pi f\tau_*}\right)\left(\frac{f}{1\ \mathrm{Hz}}\right)^{2/3}\left(\frac{\hat{\tau}}{10^6\ \mathrm{sec}}\right)^{2/3}, \qquad (11a)\end{aligned}$$

which corresponds to an amplitude

$$h \lesssim \frac{2\times 10^{-16}}{(2\pi f\tau_*)^{1/2}}\left(\frac{f}{1\ \mathrm{Hz}}\right)^{-2/3}\left(\frac{\hat{\tau}}{10^6\ \mathrm{sec}}\right)^{1/3}. \qquad (11b)$$

The factors $2\pi f\tau_*$ are of order 1 for the most abrupt bursts; slower bursts are constrained to contain the same total energy $Mc^2 = 4\pi r^2\mathscr{F}\tau_*$ and so produce a lower flux $\mathscr{F} \propto 1/\tau_*$.

B. *Galactic Sources*

Let the frequency f, event duration τ_*, and observation time $\hat{\tau}$ be given. At sufficiently high frequencies an argument identical to that for extra-galactic sources (Section IV.A) gives the same answer, but with ρ_u replaced by ρ_g: It is optimal for the bulk of the mass of the galaxy to be put into objects of mass $M \simeq (c^3/G)(2\pi f)^{-1}$, which radiate all their mass-energy Mc^2 in single bursts of duration τ_* and luminosity $L = Mc^2/\tau_*$. The strongest burst seen in time $\hat{\tau}$ has flux $\mathscr{F}$ and amplitude h at the upper limit of the inequalities

$$\mathscr{F} \lesssim \left(\frac{1}{2\pi f \tau_*}\right)\left(\frac{c^3}{\pi G^{1/3}}\right)\left(\frac{\rho_g 2\pi f \hat{\tau}}{T_H}\right)^{2/3}$$
$$\simeq \left(\frac{2 \times 10^{10}\ \text{erg/cm}^2\ \text{sec}}{2\pi f \tau_*}\right)\left(\frac{f}{1\ \text{Hz}}\right)^{2/3}\left(\frac{\hat{\tau}}{10^6\ \text{sec}}\right)^{2/3}, \tag{12a}$$

$$h \lesssim \frac{1 \times 10^{-14}}{(2\pi f \tau_*)^{1/2}}\left(\frac{f}{1\ \text{Hz}}\right)^{-2/3}\left(\frac{\hat{\tau}}{10^6\ \text{sec}}\right)^{1/3} \tag{12b}$$

for $f \gtrsim f_{\text{crit}}$ (defined below).

As one moves to lower and lower frequencies, the optimal scenario corresponds to the strongest event being farther and farther from Earth—at a distance

$$r \simeq \frac{1}{2}\left(\frac{M T_H}{\rho_g \hat{\tau}}\right)^{1/3} = \frac{1}{2}\left[\frac{(c^3/G)T_H}{\rho_g 2\pi f \hat{\tau}}\right]^{1/3}.$$

Ultimately, at critical frequency

$$f_{\text{crit}} \simeq \frac{1}{12}\frac{c^3}{G}\frac{T_H}{M_g \hat{\tau}} \simeq \frac{5\ \text{kHz}}{(\hat{\tau}/10^6\ \text{sec})}, \tag{13}$$

the distance r has grown to the galactic radius R_g. At frequencies $f < f_{\text{crit}}$, r exceeds R_g and our optimal scenario is no longer valid.

In the low-frequency regime $f < f_{\text{crit}}$, it is optimal to have just one emission event in the entire galaxy during the observation time $\hat{\tau}$, with a mean distance $r \simeq 3R_g/4$ and a luminosity $L \simeq (M_g c^2/\tau_*)(\hat{\tau}/T_H)$ so large that the entire mass of the galaxy will be exhausted in the time T_H. These events correspond to a flux and amplitude at the upper limit of the inequalities

$$\mathscr{F} \lesssim \frac{4}{9\pi}\frac{M_g c^2 \hat{\tau}}{R_g^2 T_H \tau_*} = \left(\frac{2 \times 10^9\ \text{erg/cm}^2\ \text{sec}}{2\pi f \tau_*}\right)\left(\frac{f}{1\ \text{Hz}}\right)\left(\frac{\hat{\tau}}{10^6\ \text{sec}}\right), \tag{14a}$$

$$h \lesssim \frac{2 \times 10^{-15}}{(2\pi f \tau_*)^{1/2}}\left(\frac{f}{1\ \text{Hz}}\right)^{-1/2}\left(\frac{\hat{\tau}}{10^6\ \text{sec}}\right)^{1/2} \quad \text{for } f \lesssim f_{\text{crit}}. \tag{14b}$$

Our cherished beliefs permit these events to be produced by objects of mass M anywhere in the range

$$M \lesssim \left(\frac{c^3/G}{2\pi f}\right) \simeq 4M_\odot\left(\frac{f}{f_{\text{crit}}}\right)^{-1}\left(\frac{\hat{\tau}}{10^6\ \text{sec}}\right),$$

$$M \gtrsim \frac{L\tau_*}{c^2} \simeq \frac{M_g \hat{\tau}}{T_H} \simeq 4M_\odot\left(\frac{\hat{\tau}}{10^6\ \text{sec}}\right),$$

$$M \gtrsim \left(\frac{1}{2\pi f}\right)\left(\frac{cL}{G}\right)^{1/2} \simeq \left(\frac{c^3/G}{2\pi f \tau_*}\frac{M_g \hat{\tau}}{2\pi f T_H}\right)^{1/2} \simeq \frac{4M_\odot}{(2\pi f \tau_*)^{1/2}}\left(\frac{f}{f_{\text{crit}}}\right)^{-1/2}\left(\frac{\hat{\tau}}{10^6\ \text{sec}}\right).$$

In this optimal scenario each source must experience $N \simeq Mc^2/L\tau_*$ outbursts in its lifetime. As the frequency decreases far below $f_{\rm crit}$, it ultimately reaches a limiting value

$$f_{\rm lim} \simeq \left[\frac{(c^3/G)M_g\hat{\tau}}{(2\pi M_{\rm max})^2\tau_* T_{\rm H}}\right]^{1/2} \simeq (1 \times 10^{-9}\ {\rm Hz})\left(\frac{\hat{\tau}}{\tau_*}\right)^{1/2} \qquad (15)$$

at which our optimal scenario requires source masses in excess of $M_{\rm max} = 10^8\ M_\odot$. Below this frequency the flux and amplitude limits (14) are no longer valid; but this ultralow-frequency regime is outside our domain of interest and we shall ignore it.

An attractive (albeit not highly likely) scenario for producing broadband bursts, $\tau_* \sim 1/f$, at kilohertz frequencies, with amplitude h near the upper limit (12b), (14b) is the following: It is fashionable to speculate [25] that before galaxies formed, a sizable fraction of the mass of the Universe may have condensed into massive stars ($M \sim 2$ to $20\ M_\odot$), conventionally called stars of "Population III." A significant fraction of these stars, like stars today, might have formed in close binaries which produce, after the stars have exhausted their nuclear fuel (in $\Delta t \lesssim 1$ billion years), black-hole and/or neutron-star binary systems. When our galaxy condensed out of the intergalactic medium, such binaries would have snuggled down around the galaxy [26] to form a massive halo of the type for which there is strong empirical evidence [9–11]. The orbital parameters of these compact binaries in our halo could perfectly well be such that the mean time for the two stars or holes to spiral together due to gravitational radiation reaction is of order the Hubble time $T_{\rm H}$. At the end of its inward spiral, such a binary will emit a sizable fraction of its rest mass ($\sim$2–20%) in a broadband burst of gravitational waves at kilohertz frequencies [20,27,28]. These bursts could be the events of our optimal galactic scenario.

V. Discussion

It is interesting to compare the cherished-belief upper limits of Figs. 1 and 2 with the sensitivities of gravitational-wave detectors: past, present, and future.

The first-generation Weber-type bars (1968–1976) were capable of detecting broadband bursts occurring once in $\hat{\tau} \simeq 10^6$ sec with frequencies $f \simeq 1000$ Hz and amplitudes $h \gtrsim 3 \times 10^{-16}$. This sensitivity was a little worse than our cherished-belief upper limits (Fig. 2), which explains why theorists could account for Weber's observed events [29] only by invoking unconventional hypotheses (strong beaming by sources near the galactic center [15,16,18]; or today being a very special time in the evolution of the galaxy [30]).

Second-generation detectors of the bar type and laser-interferometer type (1980–1984) are designed to have sensitivities $h \sim 10^{-18}$ for events occurring once in $\hat{\tau} \simeq 10^6$ sec with frequencies $f \simeq$ 100–1000 Hz. Such sensitivities are considerably better than our cherished-belief limits (Fig. 2). Thus, although conventional scenarios do not predict waves at this level (sensitivity worse than "type-two benchmarks"), a discovery of waves by second-generation detectors is perfectly possible within the framework of our cherished beliefs.

At much lower frequencies, $f \sim 10^{-3}$ Hz, Doppler tracking of spacecraft is being used to search for gravitational waves. The best sensitivities yet achieved, using the Viking spacecraft [31], correspond to an rms noise level $h_{\rm rms} \sim 3 \times 10^{-14}$ and a sensitivity to $\hat{\tau} = 10^6$ sec bursts of $h \sim 2 \times 10^{-13}$. These sensitivities are slightly worse than our cherished-belief limits. However, future experiments using the Solar Polar spacecraft (1985) and improved tracking technology are projected to have amplitude sensitivities a factor ~ 10 better than Viking's, and a proposed Solar Probe spacecraft (~ 1988) might do a factor ~ 100 better [32]. Such sensitivities would be somewhat better than our cherished-belief upper limits.

In conclusion, the technology of gravitational-wave detection is now crossing over our cherished-belief benchmarks. Near-future experiments will be in a realm where it is not irrational to hope for positive results!

Appendix: Notation

A. Parameters Describing the Structure of the Universe

$$T_{\rm H} = \text{Hubble time} = 1 \times 10^{10}\ \text{years} = 3 \times 10^{17}\ \text{sec}$$
$$R_{\rm H} = cT_{\rm H} = \text{Hubble radius} = 1 \times 10^{10}\ \text{lyr} = 9 \times 10^{27}\ \text{cm}$$
$$\rho_{\rm u} = \text{mean mass density of Universe} = \frac{3c^2/G}{8\pi R_{\rm H}^2} = 1 \times 10^{-8}\ M_\odot/\text{lyr}^3 = 2 \times 10^{-29}\ \text{g/cm}^3$$
$$R_{\rm g} = \text{galaxy radius} = 6 \times 10^4\ \text{lyr} = 6 \times 10^{22}\ \text{cm}$$
$$M_{\rm g} = \text{galaxy mass} = 1 \times 10^{12}\ M_\odot = 2 \times 10^{45}\ \text{g}$$
$$\rho_{\rm g} = \text{mean mass density of galaxy} = 3M_{\rm g}/4\pi R_{\rm g}^3 = 0.001 M_\odot/\text{lyr}^3 = 2 \times 10^{-24}\ \text{g/cm}^3$$
$$M_{\rm max} = \begin{pmatrix}\text{maximum mass of coherently radiating} \\ \text{object in our galaxy}\end{pmatrix} = 10^8 M_\odot$$

B. Parameters Describing Gravitational-Wave Sources and Their Radiation

$$M = \text{mass of coherently radiating source}$$
$$f = \text{mean frequency emitted by source}$$

L = luminosity of source (erg/sec) in "on" state
τ_* = "on" time for source; burst duration
N = number of "on" events during source's lifetime
n = number density of sources
r = distance to nearest source

C. *Parameters Describing Radiation Arriving at Earth*

$\hat{\tau}$ = observation time; experiment duration
$\mathscr{F}$ = flux of energy in gravitational waves (erg/cm^2 sec)
h = amplitude of gravitational waves
$\mathscr{F}_f$ = flux density of gravitational-wave background (erg/cm^2 sec Hz)
$\tilde{h}$ = square root of spectral density of amplitude of background radiation (Hz$^{-1/2}$)

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