ACM95/100b
Problem Set 8 Solutions
03/19/04

Problem 1 (10 points)

$$\begin{aligned} &y'' + \lambda^2 y = f(x) \\ &y(0) = 0 \\ &y'(1) = 1 \end{aligned} \tag{1}$$

Method 1

From the class notes of 3/5/04 we know that the solution of

$$\begin{aligned} &Ly = f \\ &B_1 y = d_1 \\ &B_2 y = d_2 \end{aligned} \tag{2}$$

will be of the form

$$y = y_1 + y_2 \tag{3}$$

where y_1 solves Ly=0 with inhomogeneous boundary conditions and y_2 solves Ly=f with homogeneous boundary conditions. What's left is to find these solutions and plug into the formula. y_1 will solve

$$\begin{aligned} &y'' + \lambda^2 y = 0 \\ &y(0) = 0 \\ &y'(1) = 1 \end{aligned} \tag{4}$$

This ODE has a general solution of the form

$$y = A e^{i \lambda x} + B e^{-i \lambda x} \tag{5}$$

The boundary conditions give

$$\begin{aligned} &A + B = 0 \\ &i \lambda (A e^{i \lambda} - B e^{-i \lambda}) = 1 \end{aligned} \tag{6}$$

The solution to this is

$$\begin{aligned} &A = \frac{-i}{2 \lambda \operatorname{Cos} \lambda} \\ &B = \frac{i}{2 \lambda \operatorname{Cos} \lambda} \end{aligned} \tag{7}$$

So

$$y_1 = \frac{\operatorname{Sin}(\lambda x)}{\lambda \operatorname{Cos} \lambda} \tag{8}$$

Now, y_2 will solve

$$\begin{aligned} &y'' + \lambda^2 y = f \\ &y(0) = 0 \\ &y'(1) = 0 \end{aligned} \tag{9}$$

We found the solution to this last week using Green's functions. So the solution is

$$y = \int_0^1 f(\xi)\, G(x|\xi)\, d\xi$$

$$G = \begin{cases} \frac{-\text{Cos}(\lambda(\xi-1))\,\text{Sin}(\lambda x)}{\lambda\,\text{Cos}\,\lambda} & 0 \le x < \xi \\ \frac{-\text{Cos}(\lambda(x-1))\,\text{Sin}(\lambda\xi)}{\lambda\,\text{Cos}\,\lambda} & \xi < x \le 1 \end{cases} \quad (10$$

So the solution to the inhomogeneous problem with inhomogeneous boundary conditions is

$$y = \frac{\text{Sin}(\lambda x)}{\lambda\,\text{Cos}\,\lambda} - \frac{\text{Cos}(\lambda(x-1))}{\lambda\,\text{Cos}\,\lambda}\int_0^x f(\xi)\,\text{Sin}(\lambda\xi)\,d\xi - \frac{\text{Sin}(\lambda x)}{\lambda\,\text{Cos}\,\lambda}\int_x^1 f(\xi)\,\text{Cos}(\lambda(\xi-1))\,d\xi \quad (11)$$

Method 2

$$\begin{aligned} &y'' + \lambda^2 y = f(x) \\ &y(0) = 0 \\ &y'(1) = 1 \end{aligned} \quad (12)$$

Make the change of dependent variable

$$z = y + a x + b \quad (13)$$

This new variable satisfies

$$\begin{aligned} &z'' + \lambda^2 z = f(x) + \lambda^2 (a x + b) \\ &z(0) = b \\ &z'(1) = 1 + a \end{aligned} \quad (14)$$

choose b=0, a=-1 and define the new function g(x)=f(x)-λ^2x.

$$\begin{aligned} &z'' + \lambda^2 z = g(x) \\ &z(0) = 0 \\ &z'(1) = 0 \end{aligned} \quad (15)$$

We found the solution to this last week using Green's functions. The solution is

$$z = \int_0^1 g(\xi)\, G(x|\xi)\, d\xi$$

$$G = \begin{cases} \frac{-\text{Cos}(\lambda(\xi-1))\,\text{Sin}(\lambda x)}{\lambda\,\text{Cos}\,\lambda} & 0 \le x < \xi \\ \frac{-\text{Cos}(\lambda(x-1))\,\text{Sin}(\lambda\xi)}{\lambda\,\text{Cos}\,\lambda} & \xi < x \le 1 \end{cases} \quad (16)$$

So the solution to the inhomogeneous problem with inhomogeneous boundary conditions is

$$y = x - \frac{\text{Cos}(\lambda(x-1))}{\lambda\,\text{Cos}\,\lambda}\int_0^x g(\xi)\,\text{Sin}(\lambda\xi)\,d\xi - \frac{\text{Sin}(\lambda x)}{\lambda\,\text{Cos}\,\lambda}\int_x^1 g(\xi)\,\text{Cos}(\lambda(\xi-1))\,d\xi \quad (17)$$

Problem 2 (5×7 points)

a)

$$\begin{aligned} &J_n = \frac{1}{2\pi}\int_0^{2\pi} e^{i(x\,\text{Sin}\,\theta - n\theta)}\, d\theta \\ &x^2 y'' + x y' + (x^2 - n^2) y = 0 \end{aligned} \quad (18)$$

Compute derivatives

$$y' = \frac{i}{2\pi}\int_0^{2\pi} \text{Sin}\,\theta\, e^{i(x\,\text{Sin}\,\theta - n\theta)}\,d\theta$$
$$y'' = \frac{-1}{2\pi}\int_0^{2\pi} \text{Sin}^2\,\theta\, e^{i(x\,\text{Sin}\,\theta - n\theta)}\,d\theta \quad (19$$

Plug in

$$x^2\,y'' + x\,y' + (x^2 - n^2)\,y = \frac{1}{2\pi}\int_0^{2\pi} (-x^2\,\text{Sin}^2\,\theta + i\,x\,\text{Sin}\,\theta + x^2 - n^2)\,e^{i(x\,\text{Sin}\,\theta - n\theta)}\,d\theta \quad (20)$$

Using a trig identity we may rewrite the integrand

$$\frac{1}{2\pi}\int_0^{2\pi} (x^2\,\text{Cos}^2\,\theta + i\,x\,\text{Sin}\,\theta - n^2)\,e^{i(x\,\text{Sin}\,\theta - n\theta)}\,d\theta \quad (21)$$

Following the hint, we set

$$u = \frac{1}{2\pi}\,e^{i(x\,\text{Sin}\,\theta - n\theta)} \quad (22)$$

and calculate

$$-u_{\theta\theta} - 2\,i\,n\,u_\theta = \frac{1}{2\pi}\,(x^2\,\text{Cos}\,\theta^2 + i\,x\,\text{Sin}\,\theta - n^2)\,e^{i(x\,\text{Sin}\,\theta - n\theta)} \quad (23)$$

This last expression is simply the integrand we found above. So we have

$$x^2\,y'' + x\,y' + (x^2 - n^2)\,y = \int_0^{2\pi} (-u_{\theta\theta} - 2\,i\,n\,u_\theta)\,d\theta =$$
$$u_\theta\,(0) - u_\theta\,(2\pi) + 2\,i\,n\,(u\,(0) - u\,(2\pi)) = \frac{i\,(x-n)}{2\pi} - \frac{i\,(x-n)}{2\pi} + 2\,i\,n\left(\frac{1}{2\pi} - \frac{1}{2\pi}\right) = 0 \quad (24)$$

b)

$$J_0\,(0) = \frac{1}{2\pi}\int_0^{2\pi} e^0\,d\theta = 1 \quad (25)$$

$$J_n\,(0) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\,n\,\theta}\,d\theta = \frac{1}{2\pi}\left(\frac{e^{-i\,n2\pi}}{-i\,n} - \frac{e^{-i\,n\,0}}{-i\,n}\right) = \frac{1}{2\pi}\left(\frac{1}{-i\,n} - \frac{1}{-i\,n}\right) = 0 \quad (26)$$

$$J_1{}'(0) = \frac{1}{2\pi}\int_0^{2\pi} i\,\text{Sin}\,\theta\,e^{-i\,\theta}\,d\theta \quad (27)$$

Breaking the complex exponential into it's real and imaginary parts gives

$$\frac{i}{2\pi}\int_0^{2\pi} \text{Sin}\,\theta\,\text{Cos}\,\theta\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} \text{Sin}^2\,\theta\,d\theta = 1/2 \quad (28)$$

$$J_n{}'(0) = \frac{1}{2\pi}\int_0^{2\pi} i\,\text{Sin}\,\theta\,e^{-in\theta}\,d\theta \quad (29)$$

Breaking the complex exponential into it's real and imaginary parts gives

$$\frac{1}{2\pi}\int_0^{2\pi} i\,\text{Sin}\,\theta\,\text{Cos}\,(n\,\theta)\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} \text{Sin}\,\theta\,\text{Sin}\,(n\,\theta)\,d\theta \quad (30)$$

By the orthogonality properties of Sine and Cosine we know that both of these integrals vanish for all $n \neq \pm 1$.

c)

Is the following true?

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n \tag{31}$$

Replace t with $e^{i\theta}$

$$e^{i x \operatorname{Sin}\theta} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{i n \theta} \tag{32}$$

Now, what we are asking is if the right side is the Fourier series of the left side on the interval $[0,2\pi]$. Since Fourier series are unique, this last question will be answered affirmatively if the Fourier coefficients of the left side are exactly $J_n(x)$. To check this, we multiply both sides by an exponential and integrate using the orthogonality property

$$\begin{gathered}\frac{1}{2\pi}\int_0^{2\pi} e^{i x \operatorname{Sin}\theta}\, e^{-i k\theta}\, d\theta = \\ \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=-\infty}^{\infty} J_n(x)\, e^{i n\theta}\, e^{-i k\theta}\, d\theta = \sum_{n=-\infty}^{\infty} J_n(x)\left(\frac{1}{2\pi}\int_0^{2\pi} e^{i n\theta}\, e^{-i k\theta}\, d\theta\right) = J_k(x)\end{gathered} \tag{33}$$

Also observe that by the definition given in part (a) we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i x \operatorname{Sin}\theta}\, e^{-i k\theta}\, d\theta = J_k(x) \tag{34}$$

So, the Fourier coefficients agree, and the expression

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n \tag{35}$$

is correct.

d)

Take the expression from part (c) and set

$$t = e^{i\theta} \tag{36}$$

We find

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{\frac{x}{2}\left(e^{i\theta}-e^{-i\theta}\right)} = e^{i x \operatorname{Sin}\theta} \tag{37}$$

So

$$e^{i x \operatorname{Sin}\theta} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{i n\theta} \tag{38}$$

Take the expression from part (c) and set

$$t = i\, e^{i\theta} \tag{39}$$

We find

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{\frac{x}{2}\left(i\, e^{i\theta}+i\, e^{-i\theta}\right)} = e^{i x \operatorname{Cos}\theta} \tag{40}$$

So

$$e^{i x \operatorname{Cos}\theta} = \sum_{n=-\infty}^{\infty} i^n\, J_n(x)\, e^{i n\theta} \tag{41}$$

e)

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n \tag{42}$$

Compute the derivative of this expression with respect to x.

$$\frac{1}{2}\left(t-\frac{1}{t}\right) e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n'(x)\, t^n \tag{43}$$

Using the generating function again gives

$$\frac{1}{2}\left(t-\frac{1}{t}\right) \sum_{n=-\infty}^{\infty} J_n(x)\, t^n = \sum_{n=-\infty}^{\infty} J_n'(x)\, t^n \tag{44}$$

Rearranging the terms on the left side gives

$$\frac{1}{2} \sum_{n=-\infty}^{\infty} J_n(x)\, t^{n+1} - \frac{1}{2} \sum_{n=-\infty}^{\infty} J_n(x)\, t^{n-1} \tag{45}$$

By shifting the indices this becomes

$$\frac{1}{2} \sum_{n=-\infty}^{\infty} J_{n-1}(x)\, t^{n} - \frac{1}{2} \sum_{n=-\infty}^{\infty} J_{n+1}(x)\, t^{n} \tag{46}$$

So we have

$$\sum_{n=-\infty}^{\infty} \left(J_{n-1}(x) - J_{n+1}(x)\right) t^n = \sum_{n=-\infty}^{\infty} 2\, J_n'(x)\, t^n \tag{47}$$

Now, as in part (a) we could let t= $e^{i\theta}$ and then make the observation that since Fourier series are unique the coefficients of both sides must be the same. This gives

$$J_{n-1} - J_{n+1} = 2\, J_n' \tag{48}$$

Problem 3 (6×6 points)

a)

$$u_t = \kappa\left(\frac{1}{r^2}\left(r^2 u_r\right)_r + \frac{1}{r^2 \operatorname{Sin}\theta}\left(\operatorname{Sin}\theta\, u_\theta\right)_\theta\right) \tag{49}$$

Let u=R(r)Ξ(θ)T(t) and simplify

$$\frac{T'}{T} = \kappa\left(\frac{1}{R\, r^2}\left(r^2 R'\right)' + \frac{1}{r^2 \operatorname{Sin}\theta} \frac{\left(\operatorname{Sin}\theta\, \Xi'\right)'}{\Xi}\right) \tag{50}$$

Since the left side is a function of t only and the right side is independent of t we have

$$\begin{aligned} &T' - \gamma \kappa T = 0 \\ &\frac{1}{R\, r^2}\left(r^2 R'\right)' + \frac{1}{r^2 \operatorname{Sin}\theta} \frac{\left(\operatorname{Sin}\theta\, \Xi'\right)'}{\Xi} = \gamma \end{aligned} \tag{51}$$

Multiplying by r^2 and simplifying gives

$$\frac{1}{\operatorname{Sin}\theta} \frac{\left(\operatorname{Sin}\theta\, \Xi'\right)'}{\Xi} = \gamma\, r^2 - \frac{1}{R}\left(r^2 R'\right)' \tag{52}$$

Since the right side is a function of r and the left side is a function of θ, both sides must be the same constant, call it -λ.

$$(r^2 R')' - (\lambda + \gamma r^2) R = 0$$
$$\text{Sin}\,\theta\,(\text{Sin}\,\theta\,\Xi')' + \lambda\,\text{Sin}^2\,\theta\;\Xi = 0 \quad (53$$

b)

If we set

$$x = \text{Cos}\,\theta \quad (54)$$

Then we have for any A(θ)=B(x(θ))

$$\text{Sin}\,\theta\,A_\theta = \text{Sin}\,\theta\,B_x\,x_\theta = -\text{Sin}^2\,\theta\,B_x = (x^2 - 1)\,B_x \quad (55)$$

Applying this result twice with $\Xi(\theta)$=y(x(θ))

$$\text{Sin}\,\theta\,(\text{Sin}\,\theta\,\Xi_\theta)_\theta = -\text{Sin}^2\,\theta\,((x^2 - 1)\,y_x)_x = \text{Sin}^2\,\theta\,((1 - x^2)\,y_x)_x \quad (56)$$

Plugging this into the ODE gives

$$\text{Sin}^2\,\theta\,(((1 - x^2)\,y_x)_x + \lambda\,y) = 0 \quad (57)$$

or

$$((1 - x^2)\,y_x)_x + \lambda\,y = 0 \quad (58)$$

c)

In general, if we have a generating function

$$g\,(r,\,x) = \sum_{n=0}^{\infty} a_n\,(x)\,r^n \quad (59)$$

Then we can compute k derivatives with respect to r

$$\frac{\partial^k}{\partial r^k}\,g\,(r,\,x) = \sum_{n=k}^{\infty} \frac{n!}{(n - k!)}\,a_n\,(x)\,r^{n-k} \quad (60)$$

and set r=0

$$\left(\frac{\partial^k}{\partial r^k}\,g\,(r,\,x)\right)_{r=0} = n!\,a_k\,(x) \quad (61)$$

We'll use this trick several times below

i)

$$\frac{1}{\sqrt{1 - 2\,r\,x + r^2}} = \sum_{n=0}^{\infty} P_n\,(x)\,r^n \quad (62)$$

Set x=1 and compute the k^{th} derivative of both sides with respect to r

$$\frac{k!}{(1 - r)^{k+1}} = \frac{d^k}{d\,r^k}\,\frac{1}{1 - r} = \frac{d^k}{d\,r^k}\,\sum_{n=0}^{\infty} P_n\,(1)\,r^n = \sum_{n=1}^{\infty} \frac{n!}{(n - k)!}\,P_n\,(1)\,r^{n-k} \quad (63)$$

Now set r=0

$$k\,! = k!\,P_k\,(1) \quad (64)$$

or

$$P_n\,(1) = 1 \quad (65)$$

ii)

$$\frac{1}{\sqrt{1-2\,r\,x+r^2}} = \sum_{n=0}^{\infty} P_n\,(x)\,r^n \tag{66}$$

Set x=-1 and compute the k^{th} derivative of both sides with respect to r

$$\frac{(-1)^k\,k!}{(1+r)^{k+1}} = \frac{d^k}{d r^k}\,\frac{1}{1+r} = \frac{d^k}{d r^k}\sum_{n=0}^{\infty} P_n\,(-1)\,r^n = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\,P_n\,(-1)\,r^{n-k} \tag{67}$$

Now set r=0

$$(-1)^k\,k\,!= k!\,P_k\,(-1) \tag{68}$$

or

$$P_n\,(-1) = (-1)^n \tag{69}$$

iii)

$$\frac{1}{\sqrt{1-2\,r\,x+r^2}} = \sum_{n=0}^{\infty} P_n\,(x)\,r^n \tag{70}$$

Set r=0

$$1 = \sum_{n=0}^{\infty} P_n\,(x)\,0^n = P_0\,(x) \tag{71}$$

$$\frac{1}{\sqrt{1-2\,r\,x+r^2}} = \sum_{n=0}^{\infty} P_n\,(x)\,r^n \tag{72}$$

Compute one derivative with respect to r.

$$\frac{x-r}{(1-2\,r\,x+r^2)^{3/2}} = \sum_{n=1}^{\infty} n\,P_n\,(x)\,r^{n-1} \tag{73}$$

Set r=0

$$x = \sum_{n=1}^{\infty} n\,P_n\,(x)\,0^{n-1} = P_1\,(x) \tag{74}$$

d)

$$\frac{1}{\sqrt{1-2\,r\,x+r^2}} = \sum_{n=0}^{\infty} P_n\,(x)\,r^n \tag{75}$$

Compute the derivative with respect to r

$$\frac{x-r}{(1-2\,r\,x+r^2)^{3/2}} = \sum_{n=1}^{\infty} n\,P_n\,(x)\,r^{n-1} \tag{76}$$

Using the generating function gives

$$\frac{x-r}{1-2\,r\,x+r^2}\sum_{n=0}^{\infty} P_n\,(x)\,r^n = \sum_{n=1}^{\infty} n\,P_n\,(x)\,r^{n-1} \tag{77}$$

Rearranging gives

$$\sum_{n=0}^{\infty} P_n(x)(x-r)\,r^n = \sum_{n=1}^{\infty} n\,P_n(x)\,(1-2\,r\,x+r^2)\,r^{n-1} \quad (78$$

Expand this into separate series

$$\sum_{n=0}^{\infty} x\,P_n(x)\,r^n - \sum_{n=0}^{\infty} P_n(x)\,r^{n+1} = \sum_{n=1}^{\infty} n\,P_n(x)\,r^{n-1} - \sum_{n=1}^{\infty} 2\,n\,x\,P_n(x)\,r^n + \sum_{n=1}^{\infty} n\,P_n(x)\,r^{n+1} = 0 \quad (79)$$

By shifting indices we combine this into a single series

$$x\,P_0(x) - P_1(x) + (3\,x\,P_1(x) - 2\,P_2(x) - P_0(x))\,r + \sum_{n=2}^{\infty} ((2\,n+1)\,x\,P_n(x) - (n+1)\,P_{n+1}(x) - n\,P_{n-1}(x))\,r^n \quad (80)$$

Taking k derivatives with respect to r and setting r=0 gives:

$$(2\,n+1)\,x\,P_k(x) - (k+1)\,P_{k+1}(x) - k\,P_{k-1}(x) = 0 \quad (81)$$

or

$$(n+1)\,P_{n+1} = (2\,n+1)\,x\,P_n - n\,P_{n-1} \quad (82)$$

Setting n=1 gives

$$P_2 = \frac{1}{2}(3\,x\,P_1 - P_0) = \frac{1}{2}(3\,x^2 - 1) \quad (83)$$

e)

$$P_n(x) = \frac{1}{2^n\,n!}\frac{d^n}{dx^n}(x^2-1)^n \quad (84)$$

Since Rodrigue's formula tells us to differentiate n times a polynomial of degree 2n the resulting polynomial, after differentiation, will be of degree n.

We can show the orthogonality relation by integrating and using Rodrigues formula:

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \frac{1}{2^n\,n!}\,\frac{1}{2^m\,m!}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n\,\frac{d^m}{dx^m}(x^2-1)^m\,dx \quad (85)$$

This integral is of the form

$$\int_{-1}^{1} u^{(n)}(x)\,v^{(m)}(x)\,dx \quad (86)$$

By integrating by parts n times we get

$$\sum_{k=1}^{n}(-1)^{k+1}\,v^{(m+k-1)}(1)\,u^{(n-k)}(1) + \sum_{k=1}^{n}(-1)^{k+1}\,v^{(m+k-1)}(-1)\,u^{(n-k)}(-1) + (-1)^n\int_{-1}^{1} u(x)\,v^{(m+n)}(x)\,dx \quad (87)$$

Notice that

$$\left(\frac{d^r}{dx^r}(x^2-1)^n\right)_{x=1} = \left(\frac{d^r}{dx^r}(x^2-1)^n\right)_{x=-1} = 0 \ \text{ for } 0 \le r < n \quad (88)$$

For this reason

$$u^{(n-k)}(1) = u^{(n-k)}(-1) = 0 \ \text{ for } k \in \{1, \ ..., n\} \quad (89)$$

and so all the boundary terms vanish. We have

$$\int_{-1}^{1} u^{(n)}(x)\, v^{(m)}(x)\, dx = (-1)^n \int_{-1}^{1} u(x)\, v^{(m+n)}(x)\, dx \tag{90}$$

(i) Suppose n>m

v is a (2m)th degree polynomial so the (m+n)th derivative of v is 0 since n>m. This means that the integral on the right is zero. Hence P_n is orthogonal to P_m for n>m. By symmetry this result also holds for n<m.

(ii) Suppose n=m

We have

$$\int_{-1}^{1} P_n(x)\, P_n(x)\, dx = \frac{1}{(2^n\, n!)^2} \int_{-1}^{1} \left(u^{(n)}\right)^2 dx \tag{91}$$

From the integration by parts procedure just used we write this as

$$\frac{(-1)^n}{(2^n\, n!)^2} \int_{-1}^{1} u(x)\, u^{(2n)}(x)\, dx = \frac{(-1)^n\, (2n)!}{(2^n\, n!)^2} \int_{-1}^{1} \left(x^2 - 1\right)^n dx \tag{92}$$

This last integral can be looked up in a table

$$\int_{-1}^{1} \left(x^2 - 1\right)^n dx = \frac{(-1)^n\, 2^{2n+1}\, (n!)^2}{(2n+1)!} \tag{93}$$

Inserting this into the integral expression gives

$$\int_{-1}^{1} P_n(x)\, P_n(x)\, dx = \frac{(-1)^n\, (2n)!}{(2^n\, n!)^2} \frac{(-1)^n\, 2^{2n+1}\, (n!)^2}{(2n+1)!} = \frac{2}{2n+1} \tag{94}$$

So we conclude

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \frac{2}{2n+1}\, \delta_{m,n} \tag{95}$$

f)

$$I = \int_{-1}^{1} x\, P_n(x)\, P_m(x)\, dx \tag{96}$$

From the recursive relation of part (d) we may rewrite this as

$$\begin{aligned} I &= \int_{-1}^{1} \left(\frac{n+1}{2n+1} P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x) \right) P_m(x)\, dx = \\ &\frac{n+1}{2n+1} \int_{-1}^{1} P_{n+1}(x)\, P_m(x)\, dx + \frac{n}{2n+1} \int_{-1}^{1} P_{n-1}(x)\, P_m(x)\, dx \end{aligned} \tag{97}$$

Using the result of part (e) these two integrals are easily found

$$\frac{n+1}{2n+1} \frac{2}{2(n+1)+1}\, \delta_{m,n+1} + \frac{n}{2n+1} \frac{2}{2(n-1)+1}\, \delta_{m,n-1} \tag{98}$$

Simplifying gives the desired result

$$I = \frac{2(n+1)}{(2n+1)(2n+3)}\, \delta_{m,n+1} + \frac{2n}{(2n+1)(2n-1)}\, \delta_{m,n-1} \tag{99}$$