

Problem Set 8 #4 Solution

Solution to PS8 Extra credit #4
ACM95b/100b

E. Sterl Phinney
12 Mar 2004

4. (7 × 3 points extra credit) Bessel Functions and FM radios

FM (Frequency Modulated) radio works by encoding an audio signal $m(t)$ (air pressure as a function of time) as frequency modulation of a radio wave about a carrier frequency f_c .

We consider the case when the audio signal is a pure tone of frequency f_M , so

$$S_{FM}(t) = A_c \cos(2\pi f_c t + \beta_f \sin(2\pi f_M t)) \quad (1)$$

β_f is known as the “frequency modulation index”.

(a) Show that $J_{-n}(x) = J_n(-x) = (-1)^n J_n(x)$.

There are many ways to show this, e.g. (i) direct inspection of the power series for the finite at $x = 0$ solutions to Bessel’s differential equation (Prob 2a eq 3), (ii) noticing that Bessel’s differential equation is invariant under parity $x \rightarrow -x$ and under $n \rightarrow -n$ and then using the initial conditions (Prob 2b eq 5) to show that J_0 is even, J_1 is odd, and then the recursion (Prob 2e, eq 8) to show that J_{2n} is even, J_{2n+1} is odd, and (iii) changing signs of x and n and the limits of integration of the integral representation (Prob 2a, eq 2).

Here we show this by (iv) using the generating function of Problem 2c, eq 6:

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(x)t^n \quad (2)$$

In eq (2), replace x by $-x$ and t by $1/t$ (if this is confusing, let $x = -\xi$ and $t = 1/\tau$ in eq (2), and then rename $\xi = x$ and $\tau = t$). The left hand side is unaffected by these changes, so (using a new dummy index m for the sum)

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{m=-\infty}^{\infty} J_m(-x)t^{-m} \quad (3)$$

Now replace the dummy index in eq (3) by $m = -n$. Since the left hand side is the same as that of eq (2), we have

$$\sum_{n=-\infty}^{\infty} J_n(x)t^n = \sum_{n=-\infty}^{\infty} J_{-n}(-x)t^n \quad (4)$$

Since power series are unique, equating the coefficients of t^n gives the first of the desired equalities $J_n(x) = J_{-n}(-x)$.

To prove the second equality, return to eq (2), and this time replace x by $-x$ and t by $-t$. Again the left hand side is unaffected by this change,

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(-x)(-t)^n = \sum_{n=-\infty}^{\infty} (-1)^n J_n(-x)t^n \quad (5)$$

Again using the fact that power series are unique, we equate the coefficients of t^n on the right sides of equations (2) and (5), we get $J_n(x) = (-1)^n J_n(-x)$. Multiplying both sides by $(-1)^n$ gives the desired equality: $J_n(-x) = (-1)^n J_n(x)$.

(b) Show that the Fourier transform of $S_{FM}(t)$ given by eq (1) is

$$\tilde{S}_{FM} = A_c/2 \sum_{n=-\infty}^{\infty} J_n(\beta_f) (\delta(f - [f_c + nf_M]) + \delta(f + [f_c + nf_M])) \quad (6)$$

Write eq (1) as

$$S_{FM}(t) = \text{Re } A_c e^{i2\pi f_c t} e^{i\beta_f \sin(2\pi f_M t)} \quad (7)$$

$$= \text{Re } A_c e^{i2\pi f_c t} \sum_{n=-\infty}^{\infty} J_n(\beta_f) e^{in2\pi f_M t}, \quad (8)$$

$$= \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta_f) \left[e^{i2\pi t(f_c + nf_M)} + e^{-i2\pi t(f_c + nf_M)} \right] \quad (9)$$

where the second equality follows from the first by substituting eq 7 of Problem 2d, and the third equality follows from the second by the fact that $\text{Re}(z) = (1/2)(z + z^*)$.

The Fourier transform is defined by

$$\tilde{S}_{FM}(f) = \int_{-\infty}^{\infty} e^{-i2\pi ft} S_{FM}(t) dt \quad (10)$$

Insert eq (9) into eq (10):

$$\tilde{S}_{FM}(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta_f) \int_{-\infty}^{\infty} \left[e^{i2\pi t(f_c + nf_M - f)} + e^{-i2\pi t(f + f_c + nf_M)} \right] dt \quad (11)$$

Recall that

$$\int_{-\infty}^{\infty} e^{i2\pi f(t-\tau)} df = \delta(t - \tau) = \delta(\tau - t) \quad (12)$$

to identify the integrals of the two terms in square brackets as delta functions. This gives the desired result,

$$\tilde{S}_{FM} = A_c/2 \sum_{n=-\infty}^{\infty} J_n(\beta_f) (\delta(f - [f_c + nf_M]) + \delta(f + [f_c + nf_M])) \quad (13)$$

(c) Plot the amplitudes of the first 10 sidebands (i.e. for $n = 0, 1, \dots, 10$) for $\beta_f = 0.3$, $\beta_F = 3$ and $\beta_f = 7$. Notice that the number of sidebands with significant amplitude is approximately $\beta_f + 1$, as one might have intuited (without the +1) from equation (21) of the PS4's problem statement.

See Figures 1-3.

(d) Show that

$$1 = \sum_{n=-\infty}^{\infty} J_n(x)^2 = J_0^2(x) + 2 \sum_{n=1}^{\infty} J_n^2(x). \quad (14)$$

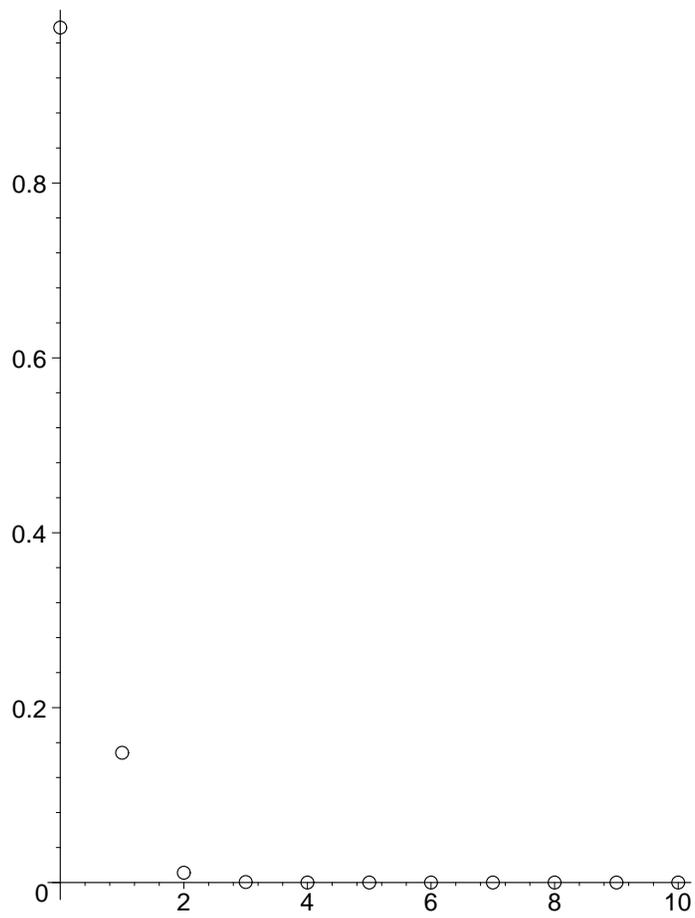


Figure 1: $J_n(\beta_f)$ versus n for $\beta_f = 0.3$

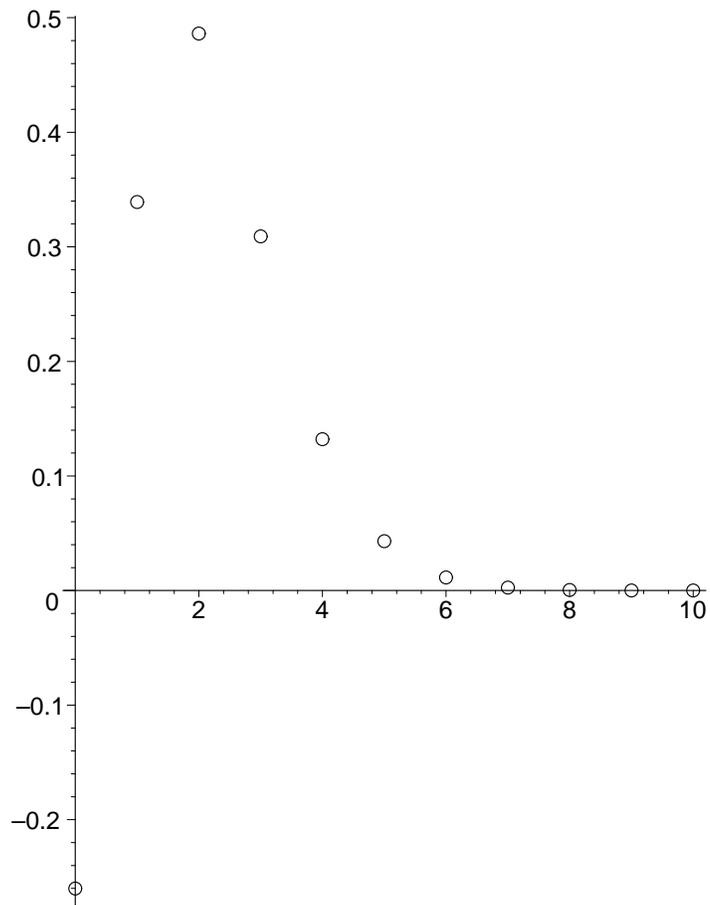


Figure 2: $J_n(\beta_f)$ versus n for $\beta_f = 3$

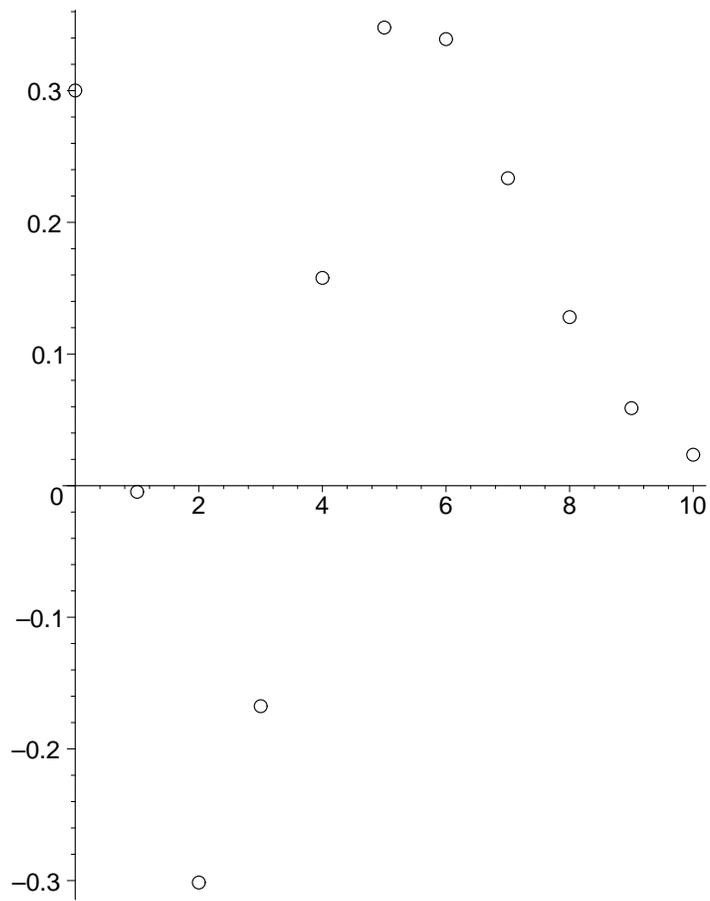


Figure 3: $J_n(\beta_f)$ versus n for $\beta_f = 7$

From eq (7) of problem 2d,

$$e^{ix \sin \theta} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\theta} \quad (15)$$

Replacing x by $-x$ in eq (15) gives

$$e^{-ix \sin \theta} = \sum_{n=-\infty}^{\infty} J_n(-x) e^{in\theta} = \sum_{n=-\infty}^{\infty} J_{-n}(x) e^{in\theta} = \sum_{m=-\infty}^{\infty} J_m(x) e^{-im\theta} \quad (16)$$

where the second equality follows from Problem 4a, and the third equality just replaces the dummy index $n = -m$.

Multiply equations (15) and (16) together to get

$$1 = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_n(x) J_m(x) e^{i(n-m)\theta} \quad (17)$$

Integrate both sides of eq (17) over θ from $\theta = 0$ to $\theta = 2\pi$, and divide by 2π :

$$1 = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_n(x) J_m(x) \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\theta} d\theta \quad (18)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} J_n(x) J_m(x) \delta_{nm} \quad (19)$$

$$= \boxed{\sum_{n=-\infty}^{\infty} J_n^2(x)} \quad (20)$$

To get the second form stated, write

$$\sum_{n=-\infty}^{\infty} J_n^2(x) = J_0^2 + \sum_{n=1}^{\infty} J_n^2(x) + \sum_{n=-1}^{-\infty} J_n^2(x) \quad (21)$$

and notice that

$$\sum_{n=-1}^{-\infty} J_n^2(x) = \sum_{n=1}^{\infty} J_{-n}^2(x) = \sum_{n=1}^{\infty} (-1)^{2n} J_n^2(x) = \sum_{n=1}^{\infty} J_n^2(x) \quad (22)$$

where in the second equality we replaced the dummy index n by $-n$, and in the third equality we used the result of part (a): $J_{-n}(x) = (-1)^n J_n(x)$.

- (e) Show that the power radiated, averaged over some very long time $2T$, $(1/2T) \int_{-T}^T S_{FM}^2 dt = A_c^2/2$. And show that this is equal to 2 times the sums of squares of the amplitudes of the sidebands.
-

Square eq (1) and use $\cos^2 \theta = 1/2 + (1/2) \cos 2\theta$:

$$S_{FM}^2(t) = \frac{A_c^2}{2} (1 + \cos[4\pi f_c t + 2\beta_f \sin(2\pi f_M t)]) \quad (23)$$

$$= \frac{A_c^2}{2} + \frac{A_c^2}{2} \operatorname{Re} e^{i[4\pi f_c t + 2\beta_f \sin(2\pi f_M t)]} \quad (24)$$

$$= \frac{A_c^2}{2} + \frac{A_c^2}{2} \operatorname{Re} e^{i4\pi f_c t} \sum_{n=-\infty}^{\infty} J_n(2\beta_f) e^{i n 2\pi f_M t} \quad (25)$$

$$= \frac{A_c^2}{2} \left(1 + \sum_{n=-\infty}^{\infty} J_n(2\beta_f) \cos[2\pi(2f_c + n f_M)t] \right) \quad (26)$$

where in the third equality we have used equation (15) [eq 7 of problem 2d]. Integrating eq (26) over time from $t = -T$ to $t = T$ and dividing by $2T$ gives:

Case 1: Provided $2f_c$ is not a multiple of f_M , so $2f_c + n f_M \neq 0$ for any n ,

$$\frac{1}{2T} \int_{-T}^T S_{FM}^2(t) dt = \frac{A_c^2}{2} \left(1 + \frac{1}{2T} \sum_{n=-\infty}^{\infty} J_n(2\beta_f) \frac{2 \sin(2\pi(2f_c + n f_M)T)}{2\pi(2f_c + n f_M)} \right). \quad (27)$$

It can be shown that for $n \gg x$

$$J_n(x) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{ex}{2n} \right)^n. \quad (28)$$

Therefore the sum in eq (27) is *extremely* rapidly convergent to a finite value. Thus as $T \rightarrow \infty$, we recover the claimed result:

$$\boxed{\frac{1}{2T} \int_{-T}^T S_{FM}^2(t) dt = \frac{A_c^2}{2}} \quad (29)$$

The fact that this is equal to 2 times the sums of squares of the amplitudes of the sidebands follows immediately from the solution to part d, equation (14) above.

Case 2: [Discovered numerically by Dave Goulet]

In the singular case when $2f_c$ is an integer multiple M of f_M , the term for $n = \boxed{-M \equiv -2f_c/f_M}$ in the sum in eq (27) should be replaced by $2T J_{-M}(2\beta_f)$, since the cosine of that term in eq (26) is then equal to one. As in case 1, all the other terms give a rapidly convergent finite sum, which vanishes when divided by $2T \rightarrow \infty$, and we get

$$\boxed{\frac{1}{2T} \int_{-T}^T S_{FM}^2(t) dt = \frac{A_c^2}{2} [1 + J_{-M}(2\beta_f)]} \quad (30)$$

Does this extra term make the stated problem wrong? No, if you are a physicist or engineer. Maybe, if you are a mathematician. Here is why: If we pick acoustic frequencies at random, only for a set of measure zero will $2f_c/f_M$ be an integer. Real radio stations do not play pure tones of infinite duration (e.g. year-long tests of the emergency broadcasting service). With a continuous frequency spectrum, the measure-zero set values of f_M that give integer $2f_c/f_M$ contribute nothing to the average power, and eq (29) holds. The Fourier transform of a pure tone of finite duration δt has a continuous

spectrum with a bandwidth $\delta f \sim 1/\delta t$, so this holds even if the radio station plays pure tones occasionally. It is also worth noting that for FM radio $88 \text{ MHz} < f < 108 \text{ MHz}$, $f_M < 10 \text{ kHz}$, and $\beta_f \sim 10$ (see below), so even for a station that played nothing but a pure tone locked to an atomic clock and tuned to be an integer fraction of a carrier frequency locked to the same atomic clock, the correction term in eq (30) has a maximum numerical value $\sim J_{-10000}(10) \simeq 10^{-29.000}$ which is a really good approximation to zero!

- (f) CD quality music is sampled at 44.1kHz, so encodes frequencies up to 22kHz. Broadcasting telephone and AM radio encodes only up to 3kHz. FM radio is intermediate in quality, so suppose it encodes frequencies up to 10kHz. We want to choose the constant k in equation (19) of the problem statement, and hence β_f as large as possible so as to maximise the dynamic range of music we can encode even when the radio signal is weak (i.e. we can measure the amplitudes of the various sidebands only with relatively poor accuracy). Show that for $f_M = 10\text{kHz}$, you have to choose $\beta_f < 8.6$ to ensure that less than 0.005 of the signal power leaks into the adjoining FM radio channels (whose sidebands start at $\pm 100\text{kHz}$ from the center channel: remember the stations are separated by 200kHz).

Since the adjacent channels start at 100 kHz from the center, with $f_M = 10 \text{ kHz}$, leakage will occur for $|n| > 10$. So we want the total power in harmonics with $n > 10$ or $n < -10$ to be less than 0.005 of the total defined by eq (14), i.e. $(1 - \sum_{-10}^{10} J_n^2(\beta_f)) < 0.005$. From Figure 4, we see that this requires $\beta_f < 8.62$. This limits the amplitude with which we can frequency modulate the signal: the peak frequency deviation can only be 86.2 kHz, not the full 100 kHz of the band.

- (g) Show that for a 3.33kHz signal encoded with the same $\beta_f = 8.6$, the fraction of the signal power leaking into the next FM radio channel is only 2×10^{-29} . Thus the owners of FM radio channels can trade between high dynamic range at low (talk show) frequencies, or lower dynamic range at high frequencies (instrument harmonics on music stations).

Since the adjacent channels start at 100 kHz from the center, with $f_M = 3.33 \text{ kHz}$, leakage will now occur for $n > 100\text{kHz}/3.33\text{kHz}=30$. So we want the total power in harmonics with $n > 30$ or $n < -30$.

It is dangerous to evaluate this as $(1 - \sum_{-30}^{30} J_n^2(\beta_f))$, because the sum is so close to one that it cannot be represented in double precision, so doing the calculation (e.g. in matlab or other ordinary floating point languages) produces pure roundoff error ($\sim 10^{-16}$ for double precision on 32-bit processors). Maple can still do the calculation this way accurately if one requests e.g. `Digits := 40`.

But a much smarter way to do the calculation, which avoids the problem of roundoff in-

troduced by subtracting two nearly equal numbers, is to write $1 - \sum_{-30}^{30} J_n^2(\beta_f) = 2 \sum_{n=31}^{\infty} J_n^2(\beta_f)$

and to recognise that the J_n are so rapidly getting tiny ($2J_{31}^2(8.62) = 2 \times 10^{-29}$, $2J_{32}^2(8.62) = 4 \times 10^{-31}$, $2J_{33}^2(8.62) = 7 \times 10^{-33}$, cf. eq (28)) that only the first one

or two terms are required to deduce the answer: $1 - \sum_{-30}^{30} J_n^2(8.62) = 2.03 \times 10^{-29}$

You can check that $\beta_f < 28$ is all that is required to keep this 3.33 kHz modulation frequency from having sidebands with more than 0.005 power outside the stations $\pm 100 \text{ kHz}$ band. This corresponds to a peak frequency deviation of 93 kHz, closer to the 100 kHz boundary than for the 10 kHz modulation frequency of the previous part.

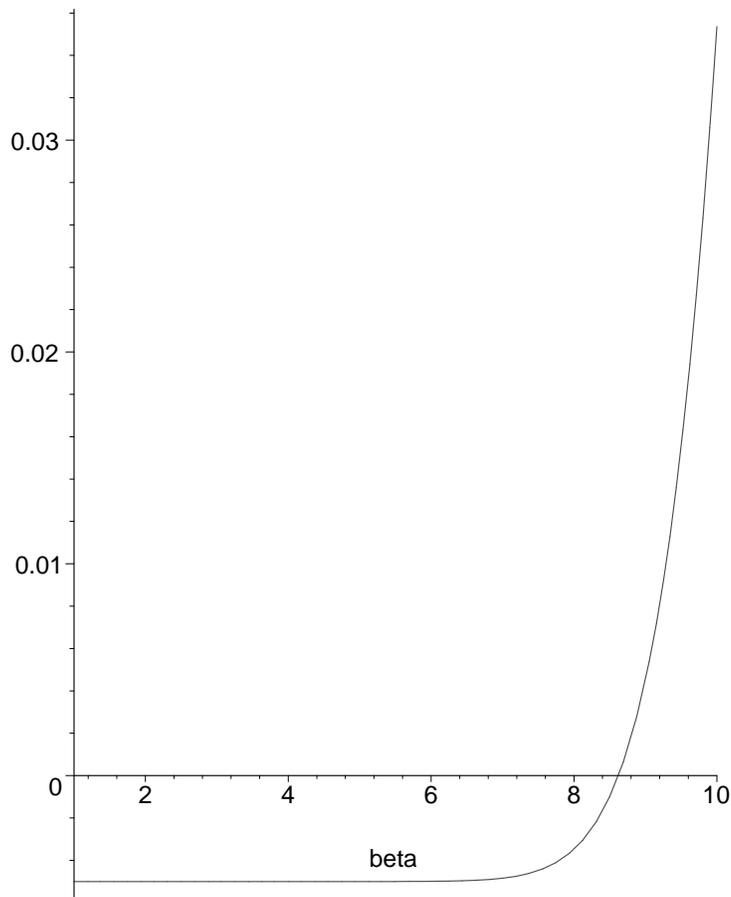


Figure 4: $(1 - \sum_{-10}^{10} J_n^2(\beta_f)) - 0.005$ versus β_f . Zero crossing (0.005 total power in sidebands with $|n| > 10$, i.e. outside allocated ± 100 kHz station frequency band) is at $\beta_f = 8.62$.