ACM95/100b
Problem Set 4 Solutions
01/29/04

Problem 1 (10 points)

Delay differential equations are important in control theory and population biology. Suppose that y(t)=$y_0(t)$ for -1≤t≤0 and subsequently satisfies

$$\frac{dy(t)}{dt} + 2y(t) - 2y(t-1) = 0, \quad t \geq 0 \tag{1}$$

Notice that the rate of change of y depends on its value at a previous time (e.g. rate of population growth depends on the number of babies born 18-36 years ago; because of finite neuron speed and processing time, you brake or accelerate your car about 1/10 second after the visual inputs motivating the changes).
Show that the Laplace transform of y(t) satisfies:

$$Y(s) = \frac{y_0(0) + 2e^{-s}\int_{-1}^{0} e^{-st}\, y_0(t)\, dt}{2 + s - 2e^{-s}} \tag{2}$$

Solution to Problem 1

$$\begin{aligned} &\frac{dy(t)}{dt} + 2y(t) - 2y(t-1) = 0, \quad t \geq 0 \\ &y(t) = y_0(t) \text{ for } -1 \leq t \leq 0 \end{aligned} \tag{3}$$

Transform each part of the equation:

$$\begin{aligned} &\mathcal{L}[y(t)] = Y \\ &\mathcal{L}\left[\frac{dy(t)}{dt}\right] = sY - y(0) = sY - y_0(0) \\ &\mathcal{L}[y(t-1)] = \\ &\quad \int_0^\infty y(t-1)\, e^{-st}\, dt = e^{-s}\left(\int_{-1}^{0} y(x)\, e^{-sx}\, dx + \int_0^\infty y(x)\, e^{-sx}\, dx\right) = e^{-s}\left(\int_{-1}^{0} y_0(x)\, e^{-sx}\, dx + Y\right) \end{aligned} \tag{4}$$

Plugging these into the equation gives:

$$(s + 2 - 2e^{-s})\,Y - y_0(0) - 2e^{-s}\int_{-1}^{0} y_0(x)\, e^{-sx}\, dx = 0 \tag{5}$$

Solving for Y gives:

$$Y = \frac{y_0(0) + 2e^{-s}\int_{-1}^{0} y_0(x)\, e^{-sx}\, dx}{2 + s - 2e^{-s}} \tag{6}$$

Problem 2 (2×7 points)

Let f(x) be periodic, with period L, so that f(x+L)=f(x). Define

$$g(x) = \begin{cases} f(x) & \text{for } 0 < x < L \\ 0 & \text{for } x > L \end{cases} \tag{7}$$

and let G(s) be the transform of g(x).

a) Show that the Laplace transform of f(x) is

$$F(s) = \frac{G(s)}{1 - e^{-sL}} \text{ for } s > 0 \tag{8}$$

b) Use the result of (a) to find the Laplace transform of the square wave of period L defined by

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < L/2 \\ 0 & \text{for } L/2 < x < L \end{cases} \tag{9}$$

Solution to Problem 2

a)

Laplace transform f(x) from the definition and break the integration interval into infinitely many intervals of length L

$$F(s) = \mathcal{L}[f] = \int_0^\infty f(t)\, e^{-st}\, dt = \sum_{n=0}^{\infty} \int_{nL}^{(n+1)L} f(t)\, e^{-st}\, dt \tag{10}$$

Now make the change of variable in each integral t=x+nL and use the fact that f(x+nL)=f(x)

$$\sum_{n=0}^{\infty} \int_0^L f(x + nL)\, e^{-s(x+nL)}\, dx = \int_0^L f(x)\, e^{-sx}\, dx \sum_{n=0}^{\infty} e^{-snL} \tag{11}$$

If Re(s)>0 the remaining sum is a convergent geometric series whose sum we know exactly

$$\frac{1}{1 - e^{-sL}} \int_0^L f(x)\, e^{-sx}\, dx \tag{12}$$

Now transform g(x) from the definition:

$$G(s) = \mathcal{L}[g] = \int_0^\infty g(x)\, e^{-sx}\, dx = \int_0^L f(x)\, e^{-sx}\, dx \tag{13}$$

Observe that these two results give

$$F(s) = \frac{G(s)}{1 - e^{-sL}} \text{ for } \mathrm{Re}(s) > 0 \tag{14}$$

b)

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < L/2 \\ 0 & \text{for } L/2 < x < L \end{cases} \tag{15}$$

The corresponding g(x) is defined by:

$$g(x) = \begin{matrix} f(x) & \text{for } 0 < x < L \\ 0 & \text{for } x > L \end{matrix} = \begin{matrix} 1 & \text{for } 0 < x < L/2 \\ 0 & \text{for } x > L/2 \end{matrix} \quad (16$$

$$G(s) = \mathcal{L}[g] = \int_0^\infty g(x)\, e^{-s x}\, dx = \int_0^{L/2} e^{-s x}\, dx = \frac{1 - e^{-s L/2}}{s} \quad (17)$$

Plugging this into the formula in part (a) gives

$$F(s) = \frac{\frac{1-e^{-s L/2}}{s}}{1 - e^{-s L}} = \frac{1 - e^{-s L/2}}{s\,(1 - e^{-s L})} = \frac{1}{s\,(1 + e^{-s L/2})} \quad (18)$$

An alternative way to do this problem is to notice that

$$f(x) = \sum_{n=0}^{\infty} (-1)^n\, H\left(x - \frac{n L}{2}\right) \quad (19)$$

Then transform term by term using a shifting theorem and the known sum of a geometric series

$$F(s) = \sum_{n=0}^{\infty} (-1)^n\, \frac{e^{-s n L/2}}{s} = \frac{1}{s\,(1 + e^{-s L/2})} \quad (20)$$

Problem 3 (15 points)

The Heaviside expansion theorem. If a transform F(s) can be written as a ratio

$$F(s) = \frac{G(s)}{H(s)} \quad (21)$$

where G(s) and H(s) are analytic functions, and H(s) has only simple, isolated zeros at $s = s_k$.

a) (8 points) Show that the inverse transform is for sufficiently large t (see part(b))

$$f(t) = \mathcal{L}^{-1}\left\{\frac{G(s)}{H(s)}\right\} = \sum_k \frac{G(s_k)}{H'(s_k)}\, e^{s_k t} \quad (22)$$

b) (4 points) Consider G(s)=exp(-3s), H(s)=s-2. For what t does the theorem of part (a) apply and why? [Hint: consider Mellin inversion contours]

c) (3 points) Give a more interesting example of the use of this result (one was given surreptitiously in class, which you may use if you recognize it, or you may invent one yourself).

Solution to Problem 3

(a)

$$f(t) = \mathcal{L}^{-1}\left\{\frac{G(s)}{H(s)}\right\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{G(s)}{H(s)}\, e^{s t}\, ds \quad (23)$$

We will assume that f(t) is a piecewise continuous function of exponential order. As shown in previous problem sets and in the class notes, the transform of such a function, G/H, will →0 as Re(s)→∞. It will be helpful to extend this and assume that G/H→0 as |s|→∞.
Assume that either H has finitely many roots or that if it has infinitely many, the real parts of those roots are bounded from above. This allows us to pick c to the right of all the roots of H(s). Then use the following contour which we call Γ_R.

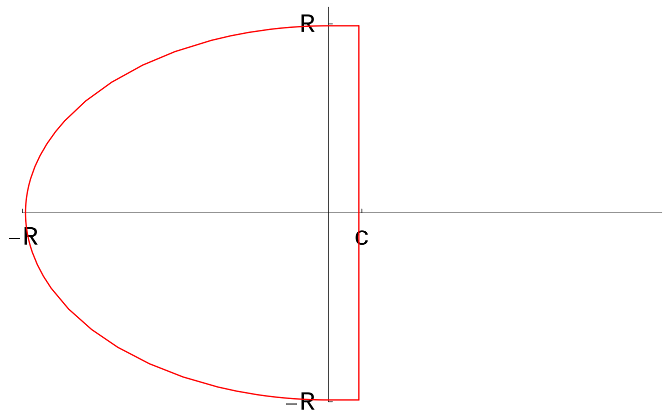

Figure 1

By the Residue Theorem we have:

$$2\pi i \sum_k \text{Res}\left(\frac{G(s)}{H(s)} e^{st}, s_k\right) = \lim_{R\to\infty} \int_{\Gamma_R} \frac{G(s)}{H(s)} e^{st}\, ds = \tag{24}$$
$$\lim_{R\to\infty}\left(\int_{c-iR}^{c+iR} \frac{G(s)}{H(s)} e^{st}\, ds + \int_{c+iR}^{iR} \frac{G(s)}{H(s)} e^{st}\, ds + \int_{-iR}^{c-iR} \frac{G(s)}{H(s)} e^{st}\, ds + \int_{C_R} \frac{G(s)}{H(s)} e^{st}\, ds\right)$$

The horizontal portions vanish as R→∞. This is easily seen from the following bound:

$$\left|\int_{c+iR}^{iR} \frac{G(s)}{H(s)} e^{st}\, ds\right| \le c\, e^{ct} \underset{s\in(iR,c+iR)}{\text{Max}} \left|\frac{G(s)}{H(s)}\right| \tag{25}$$

We have previously assumed that G/H→∞ so this maximum vanishes. We do likewise for the other straight horizontal part of the contour. For the semicircular part we have the following bound

$$\left|\int_{C_R} \frac{G(s)}{H(s)} e^{st}\, ds\right| = \left|\int_{\pi/2}^{3\pi/2} \frac{G(Re^{i\theta})}{H(Re^{i\theta})} R\, i\, e^{i\theta} e^{Re^{i\theta} t}\, d\theta\right| \le R \underset{C_R}{\text{Max}} \left|\frac{G}{H}\right| \int_{\pi/2}^{3\pi/2} e^{R t \cos\theta}\, d\theta \tag{26}$$

Since Cosine is non-positive on this interval and since the maximum of G/H is assumed to vanish, this integral vanishes by Jordan's Lemma. We conclude:

$$\lim_{R\to\infty} \int_{c-iR}^{c+iR} \frac{G(s)}{H(s)} e^{st}\, ds = 2\pi i \sum_k \text{Res}\left(\frac{G}{H} e^{st}, s_k\right) \tag{27}$$

Since the roots of H are simple we have

$$\text{Res}\left(\frac{G}{H} e^{s t}, s_k\right) = \lim_{s \to s_k} \frac{(s - s_k) G(s) e^{s t}}{H(s)} = \frac{G(s_k) e^{s_k t}}{H'(s_k)} \tag{28}$$

by L'Hopital's rule. Hence:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{G(s)}{H(s)}\right\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{G(s)}{H(s)} e^{s t} \, ds = \sum_k \frac{G(s_k)}{H'(s_k)} e^{s_k t} \tag{29}$$

Extending this result to cover the function found in the next part of the problem.
Note that our result still holds even if G/H were equal to some function with an exponential in it: $(P/Q)\, e^{-a s}$ where P/Q vanishes for large s. G/H is no longer a function which vanishes for large |s| on the contour C_R unless we put certain restrictions on t. Reconsider the bound on the semicircular part:

$$\left|\int_{C_R} \frac{P(s)}{Q(s)} e^{s(t-a)} \, ds\right| = \left|\int_{\pi/2}^{3\pi/2} \frac{P(R e^{i\theta})}{Q(R e^{i\theta})} R\, i\, e^{i\theta} e^{R e^{i\theta}(t-a)} \, d\theta\right| \leq R \,\text{Max}\left|\frac{P}{Q}\right| \int_{\pi/2}^{3\pi/2} e^{R(t-a)\cos\theta} \, d\theta \tag{30}$$

By reasons explained previously, if we require t>a, Jordan's lemma applies. Now reconsider the bound on the horizontal parts

$$\left|\int_{c+iR}^{iR} \frac{P(s)}{Q(s)} e^{s(t-a)} \, ds\right| \leq c\, e^{c(t-a)} \underset{s \in (iR, c+iR)}{\text{Max}} \left|\frac{P(s)}{Q(s)}\right| \tag{31}$$

Again, this vanishes. So, if G/H is of the form

$$\frac{G(s)}{H(s)} = \sum_{n=1}^{N} \frac{P_i(s)}{Q_i(s)} e^{-a_i s} \tag{32}$$

With P_i / Q_i vanishing for large s and Q_i having only simple roots all of which have real parts bounded above. Then our result is still valid for all t >Max(a_i).

$$f(t) = \sum_k \frac{G(s_k)}{H'(s_k)} e^{s_k t} \quad \text{for } t > \text{Max}(a_i) \tag{33}$$

b)

Method 1

Part (a) tells us that

$$f(t) \sim \frac{G(s_1)}{H'(s_1)} e^{s_1 t} = \frac{e^{-6}}{1} e^{2t} = e^{2t-6} \text{ for } t > 3 \tag{34}$$

Method 2

From the shifting theorems we know:

$$\mathcal{L}^{-1}\left[\frac{e^{-3s}}{s-2}\right] = H(t-3)\, e^{2(t-3)} \tag{35}$$

So our estimate in part (a) is only valid for t>3.

Method 3

Suppose we try to invert the function using the Mellin inversion formula:

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{-3s}}{s-2}\, e^{st}\, ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{s-2}\, e^{s(t-3)}\, ds \tag{36}$$

Where c>2. We use the following contour:

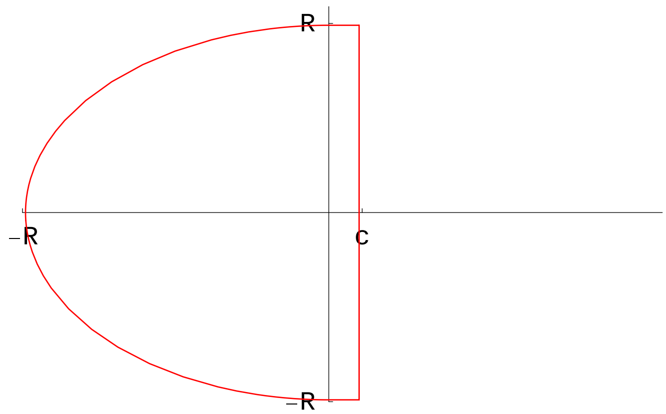

Figure 2

The residue theorem gives:

$$\int_{\Gamma_R} \frac{1}{s-2}\, e^{s(t-3)}\, ds = 2\pi i \operatorname{Res}\left(\frac{1}{s-2}\, e^{s(t-3)}, 2\right) = 2\pi i\, e^{2t-6} \tag{37}$$

The integral on each of the horizontal parts vanishes. For example

$$\left|\int_{c+ir}^{iR} \frac{1}{s-2}\, e^{s(t-3)}\, ds\right| = \left|\int_{c}^{0} \frac{1}{x+iR-2}\, e^{(x+ir)(t-3)}\, dx\right| \leq \int_{0}^{c} \frac{1}{|z+iR-2|}\, e^{z(t-3)}\, dz \to 0 \tag{38}$$

The integral on the semicircular part of the contour can be bounded:

$$\left|\int_{C_R} \frac{1}{s-2} e^{s(t-3)} \, ds\right| = \left|\int_{\pi/2}^{3\pi/2} \frac{i R e^{i\theta}}{R e^{i\theta} - 2} e^{R e^{i\theta}(t-3)} \, d\theta\right| \leq$$
$$\int_{\pi/2}^{3\pi/2} \frac{R}{|R e^{i\theta} - 2|} e^{R(t-3)\cos\theta} \, d\theta \leq \int_{\pi/2}^{3\pi/2} \frac{R}{R-2} e^{R(t-3)\cos\theta} \, d\theta \quad (39$$

Since Cos θ is negative in this interval, the integral will vanish by Jordan's lemma provided that t>3. So for t>3 we have:

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{-3s}}{s-2} e^{st} \, ds = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{1}{s-2} e^{s(t-3)} \, ds = e^{2(t-3)} \quad (40)$$

For t<3 we would use this contour

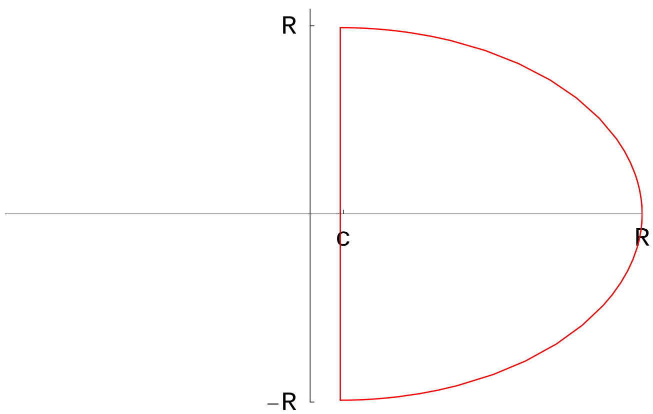

Figure 3

and we would find f(t)=0 since there are no poles inside and the circular part would vanish by Jordan's lemma. So we conclude:

$$f(t) = H(t-3) \, e^{2(t-3)} \quad (41)$$

And we see that the estimate in part (a) only holds for t>3.

c) (3 points) Give a more interesting example of the use of this result (one was given surreptitiously in class, which you may use if you recognize it, or you may invent one yourself).

Example from class

$$F(s) = \frac{1}{s(s + \lambda e^{-s})} \quad (42)$$

As shown in class, the number and location of the roots of the denominator depends on the real parameter λ. Letting s=a+bi we get two equations for the locations of the roots:

$$a + \lambda e^{-a} \cos b = 0$$
$$b - \lambda e^{-a} \sin b = 0 \quad (43)$$

When a>Max(1,ln|λ|) the first equation can't have a solution since

$$|\lambda\, e^{-a} \operatorname{Cos} b| \le |\lambda|\, e^{-a} < 1 < a \tag{44}$$

This means that all the roots are confined to $\mathrm{Re}(s) \le \mathrm{Max}(1, \ln|\lambda|)$. In addition $F(s) \to 0$ for large s, so the result we proved in part (a) applies to the inversion of this function:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{\lambda} + \sum_k \frac{G(s_k)}{H'(s_k)} e^{s_k t} = \frac{1}{\lambda} + \sum_k \frac{1}{s_k(1+s_k)} e^{s_k t} \tag{45}$$

Problem 4 (6×4 points)

Find solutions, if they exist, for the following boundary value problems:

$$\begin{aligned}
&\text{(a)}\ y'' + \pi^2 y = 0, \quad y(0) = 2, \quad y(1) = 0\\
&\text{(b)}\ y'' + \pi^2 y = 0, \quad y(0) = 1, \quad y(1) = 2\\
&\text{(c)}\ y'' + 9y = 0, \quad y(0) = 0, \quad y(\pi) = 0\\
&\text{(d)}\ y'' + 9y = x, \quad y(0) = 0, \quad y(\pi) = 0\\
&\text{(e)}\ y'' + 9y = \operatorname{Sin} x, \quad y(0) = 0, \quad y(\pi) = 1\\
&\text{(f)}\ y'' + 9y = \operatorname{Sin} x, \quad y(0) = 1, \quad y(\pi) = -1
\end{aligned} \tag{46}$$

Solution to Problem 4

Each of these boundary value problems is of the form:

$$\begin{aligned}
&y'' + a^2 y = f(x)\\
&y(0) = A, \quad y(L) = B
\end{aligned} \tag{47}$$

The homogeneous ODE will have solutions of the form $e^{r x}$. Plugging this in gives two imaginary values for $r = \pm i a$ which lead us to a homogeneous solution of the form:

$$y_h = \alpha \operatorname{Sin}(a x) + \beta \operatorname{Cos}(a x) \tag{48}$$

Particular solutions are most easily found via the method of undetermined coefficients.

(a)

The ODE is homogeneous, so the general solution is of the form:

$$y = y_h = \alpha \operatorname{Sin}(\pi x) + \beta \operatorname{Cos}(\pi x) \tag{49}$$

The boundary conditions require:

$$\begin{aligned}
\beta &= 2\\
\beta &= 0
\end{aligned} \tag{50}$$

So there is no solution.

(b)

The ODE is homogeneous, so the general solution is of the form:

$$y = y_h = \alpha \operatorname{Sin}(\pi x) + \beta \operatorname{Cos}(\pi x) \quad (51$$

The boundary conditions require:

$$\begin{aligned} \beta &= 1 \\ \beta &= -2 \end{aligned} \quad (52)$$

So there is no solution.

(c)

The ODE is homogeneous, so the general solution is of the form:

$$y = y_h = \alpha \operatorname{Sin}(3 x) + \beta \operatorname{Cos}(3 x) \quad (53)$$

The boundary conditions require:

$$\begin{aligned} \beta &= 0 \\ \beta &= 0 \end{aligned} \quad (54)$$

So for any α, there is a solution of the form:

$$y = \alpha \operatorname{Sin}(\pi x) \quad (55)$$

(d)

The ODE is inhomogeneous, so the general solution is of the form:

$$y = y_h + y_p = \alpha \operatorname{Sin}(3 x) + \beta \operatorname{Cos}(3 x) + y_p \quad (56)$$

We find a particular solution by the method of undetermined coefficients (see problem set 1)

$$y_p = a + b x \quad (57)$$

Plug this into the ODE to get

$$x = (a + b x)'' + 9 (a + b x) = 9 a + 9 b x \quad (58)$$

We see that a=0 and b=1/9. So the general solution is

$$y = \alpha \operatorname{Sin}(3 x) + \beta \operatorname{Cos}(3 x) + x / 9 \quad (59)$$

The boundary conditions require:

$$\begin{aligned} \beta &= 0 \\ \beta &= \pi / 9 \end{aligned} \quad (60)$$

So there is no solution.

(e)

The ODE is inhomogeneous, so the general solution is of the form:

$$y = y_h + y_p = \alpha \, \text{Sin}(3\,x) + \beta \, \text{Cos}(3\,x) + y_p \tag{61}$$

We find a particular solution by the method of undetermined coefficients (see problem set 1)

$$y_p = a \, \text{Sin}\, x + b \, \text{Cos}\, x \tag{62}$$

Plug this into the ODE to get:

$$\text{Sin}\, x = (a \, \text{Sin}\, x + b \, \text{Cos}\, x)'' + 9\,(a \, \text{Sin}\, x + b \, \text{Cos}\, x) = 8\,a \, \text{Sin}\, x + 8\,b \, \text{Cos}\, x \tag{63}$$

We see that b=0 and a=1/8, so the general solution is of the form:

$$y = \alpha \, \text{Sin}(3\,x) + \beta \, \text{Cos}(3\,x) + \frac{1}{8} \, \text{Sin}\, x \tag{64}$$

The boundary conditions require:

$$\begin{aligned} \beta &= 0 \\ \beta &= -1 \end{aligned} \tag{65}$$

So there is no solution.

(f)

From part (e) we know that the solution will be of the form

$$y = \alpha \, \text{Sin}(3\,x) + \beta \, \text{Cos}(3\,x) + \frac{1}{8} \, \text{Sin}\, x \tag{66}$$

The boundary conditions require:

$$\begin{aligned} \beta &= 1 \\ \beta &= 1 \end{aligned} \tag{67}$$

So for any α, there is a solution of the form

$$y = \alpha \, \text{Sin}(3\,x) + \text{Cos}(3\,x) + \frac{1}{8} \, \text{Sin}\, x \tag{68}$$

Problem 5 (8 points)

Discuss the consistency of your answers to problem 4a-f with the two general theorems stated below:

I. A linear second order ODE with nonhomogeneous boundary conditions has a unique solution if and only if the corresponding homogeneous problem has only the trivial solution.

II. If the homogeneous boundary condition problem has a nontrivial solution, the corresponding nonhomogeneous boundary value problem either has no solution or an infinity of solutions given by y=y_p+c u where y_p is a particular solution of the nonhomogeneous problem, c is an arbitrary constant and u is a nontrivial solution of the homogeneous boundary problem.

Solution to Problem 5

The homogeneous boundary value problem

$$y'' + \pi^2 y = 0, \quad y(0) = 0, \quad y(1) = 0 \tag{69}$$

has the non-trivial solution y= Sin πx. Theorem I says that BVPs (a) and (b) will not have a unique solution. Theorem II tells us that (a) and (b) will either have no solution or infinitely many. We found this to be the case, since we showed that neither (a) nor (b) has a solution.

The homogeneous BVP

$$y'' + 9y = 0, \quad y(0) = 0, \quad y(\pi) = 0 \tag{70}$$

has the non-trivial solution y= Sin3x as we found in part (c). Consider again part (d) where the ODE had an inhomogeneous term.

$$y'' + 9y = x, \quad y(0) = 0, \quad y(\pi) = 0 \tag{71}$$

If we write y=z+x/9 we get

$$z'' + 9z = 0, \quad z(0) = 0, \quad z(\pi) = -\pi/9 \tag{72}$$

In this form, theorems I and II apply. Make similar changes in parts (e) and (f) by letting y=z+(Sin x)/8:

$$\begin{aligned} &(e)\ z'' + 9z = 0, \quad z(0) = 0, \quad z(\pi) = 1 \\ &(f)\ z'' + 9z = 0, \quad z(0) = 1, \quad y(\pi) = -1 \end{aligned} \tag{73}$$

With parts (d)-(f) written this way, the results of part (c) together with theorems I and II tells us that these problems will have either no solution or infinitely many. As we found, (d) and (e) have no solution while (f) has infinitely many solutions of the form:

$$y = \alpha \operatorname{Sin}(3x) + \operatorname{Cos}(3x) + \frac{1}{8} \operatorname{Sin} x \tag{74}$$

for any α.

Problem 6 (10 points)

Let λ_i be the eigenvalues and y_i be the corresponding eigenfunctions of a linear differential operator L. Show that a solution of

$$L y + \lambda y = \sum_{k=1}^{n} A_k y_k \tag{75}$$

where the A_k are given constants is

$$y = \sum_{k=1}^{n} \frac{A_k y_k}{\lambda - \lambda_k} \tag{76}$$

provided that λ is not an eigenvalue.

Solution to Problem 6

We know that

$$L\, y_k = -\lambda_k\, y_k \tag{77}$$

Applying L to the proposed solution we find:

$$L\, y = L \sum_{k=1}^{n} \frac{A_k\, y_k}{\lambda - \lambda_k} = \sum_{k=1}^{n} \frac{A_k\, L\, y_k}{\lambda - \lambda_k} = -\sum_{k=1}^{n} \frac{A_k\, \lambda_k\, y_k}{\lambda - \lambda_k} \tag{78}$$

The operator passes through the sum because of it's linearity. Assuming that $\lambda \neq \lambda_k$ we have:

$$L\, y + \lambda\, y = -\sum_{k=1}^{n} \frac{A_k\, \lambda_k\, y_k}{\lambda - \lambda_k} + \sum_{k=1}^{n} \frac{A_k\, \lambda\, y_k}{\lambda - \lambda_k} = \sum_{k=1}^{n} \frac{A_k\, y_k}{\lambda - \lambda_k} (\lambda - \lambda_k) = \sum_{k=1}^{n} A_k\, y_k \tag{79}$$

Problem 7 (10 points)

Solve $y''+\lambda y = \text{Sin } \pi x + 2 \text{ Sin } 2\pi x + 3 \text{ Sin } 3\pi x$, $y(0)=y(1)=0$, by using the method of problem 6. What if any restriction on λ is required so that this solution be valid?

Solution to Problem 7

First we find the eigenvalues and eigenfunctions for L

$$\begin{aligned} & y_k'' + \lambda_k\, y_k = 0 \\ & y_k(0) = y_k(1) = 0 \end{aligned} \tag{80}$$

We look for a solution of the form

$$y = e^{rx} \tag{81}$$

Plug this into the ODE

$$r^2 + \lambda_k = 0 \tag{82}$$

So we have two solutions

$$y = A\, e^{i (\lambda_k)^{1/2} x} + B\, e^{-i (\lambda_k)^{1/2} x} \tag{83}$$

The boundary conditions require:

$$\begin{aligned} & A + B = 0 \\ & A\, e^{i (\lambda_k)^{1/2}} + B\, e^{-i (\lambda_k)^{1/2}} = 0 \end{aligned} \tag{84}$$

This linear system only has non-trivial solutions if the determinant is zero, otherwise the matrix would be invertible and we would find A=B=0 which doesn't give an eigenfunction.

$$0 = \text{Det} \begin{pmatrix} 1 & 1 \\ e^{i (\lambda_k)^{1/2}} & e^{-i (\lambda_k)^{1/2}} \end{pmatrix} = e^{-i (\lambda_k)^{1/2}} - e^{i (\lambda_k)^{1/2}} \tag{85}$$

Solving this for λ_k gives:

$$\lambda_k = (n\pi)^2 \quad (86$$

The value n=0 just gives the trivial solution y=0, so the eigenvalues are all positive. (This could also have been realized by recognizing this as a regular Sturm-Liouville eigenvalue problem and using the Rayleigh quotient). Knowing the eigenvalues allows us to find A and B. Doing this gives the general solution

$$\begin{aligned} \lambda_k &= k^2 \pi^2 \\ y_k &= \text{Sin}(k\pi x) \end{aligned} \tag{87}$$

So the problem that we want to solve is:

$$\begin{aligned} &y'' + \lambda y = y_1 + 2 y_2 + 3 y_3 \\ &y(0) = y(1) = 0 \end{aligned} \tag{88}$$

From problem 6 the solution be of the form:

$$y = \sum_{k=1}^{3} \frac{A_k y_k}{\lambda - \lambda_k} = \frac{\text{Sin}(\pi x)}{\lambda - \pi^2} + \frac{2\,\text{Sin}(2\pi x)}{\lambda - 4\pi^2} + \frac{3\,\text{Sin}(3\pi x)}{\lambda - 9\pi^2} \tag{89}$$

Obviously we must require $\sqrt{\lambda} \neq \pi, 2\pi, 3\pi$, or else the result of problem 6 doesn't apply (i.e. the solution is more complicated with resonant terms). What if λ is one of the other eigenvalues, say $\lambda = n^2 \pi^2$ for n>3? You can check that in this case there will be solutions of the form:

$$y = \frac{\text{Sin}(\pi x)}{(n^2 - 1)\pi^2} + \frac{2\,\text{Sin}(2\pi x)}{(n^2 - 4)\pi^2} + \frac{3\,\text{Sin}(3\pi x)}{(n^2 - 9)\pi^2} + A\,\text{Sin}(n\pi x) \tag{90}$$

for any value of A. So if the formula of problem 6 is to give the entire solution, we must require that λ not be any of the eigenvalues, not just the eigenvalues corresponding to the eigenfunctions found in the inhomogeneous part of the equation.