Problem Set 4

January 30, 2004	Due Feb 6, 2004
ACM 95b/100b	3pm in Firestone 303
E. Sterl Phinney	(2 pts) Include grading section number

Useful reading for problem 1: 1/28/2004 Lecture notes. For Problems 2 and 3: Arfken Chapter on Integral Transforms, Laplace transform sections. For Problems 4-7: Carrier and Pearson Chapters 6 and 7 or Arfken Chapter on Sturm-Liouville Theory (overkill).

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1. (10 points) Delay-differential equations are important in control theory and population biology. Suppose that $y(t) = y_0(t)$ for $-1 \leq t \leq 0$ and subsequently $y(t)$ satisfies

$$\frac{dy(t)}{dt} + 2y(t) - 2y(t-1) = 0, \quad t \geq 0\ . \tag{1}$$

Notice that the rate of change of y depends on its value at a previous time (e.g. rate of population growth depends on number of babies born 18-36 years ago; because of finite neuron signal speed and processing time, you brake or accelerate your car about 1/10 second *after* the visual inputs motivating the changes).

a) Show that the Laplace transform of $y(t)$ satisfies

$$Y(s) = \frac{y_0(0) + 2e^{-s}\int_{-1}^{0} e^{-st} y_0(t)\, dt}{2 + s - 2e^{-s}}\ . \tag{2}$$

2. (2×7 points) Let $f(x)$ be periodic, with period L, so that $f(x + L) = f(x)$. Define

$$g(x) = \begin{cases} f(x) & \text{for } 0 < x < L, \\ 0 & \text{for } x > L \end{cases} \tag{3}$$

and let $G(s)$ be the Laplace transform of $g(x)$.

a) Show that the Laplace transform of $f(x)$ is

$$F(s) = \frac{G(s)}{1 - e^{-sL}} \quad \text{for } s > 0\ . \tag{4}$$

b) Use the result of (a) to find the Laplace transform of the square wave of period L defined by

$$f(x) = \begin{cases} 1 & \text{for } 0 < x < L/2, \\ 0 & \text{for } L/2 < x < L \end{cases} \tag{5}$$

3. (15 points) The Heaviside expansion theorem. If a transform $F(s)$ can be written as a ratio

$$F(s) = \frac{G(s)}{H(s)} \tag{6}$$

where $G(s)$ and $H(s)$ are analytic functions, and $H(s)$ has only simple, isolated zeros at $s = s_k$,

a) (8 points) Show that the inverse transform is for suitably large t (see part (b))

$$f(t) = \mathcal{L}^{-1}\left\{\frac{G(s)}{H(s)}\right\} = \sum_k \frac{G(s_k)}{H'(s_k)} e^{s_k t} \tag{7}$$

b) (4 points) Consider $G(s) = \exp(-3s)$, $H(s) = s - 2$. For what t does the theorem of part (a) apply and why? [Hint: consider Mellin inversion contours].

c) (3 points) Give a more interesting example of the use of this result (one was given surreptitiously in class, which you may reuse if you recognise it, or you may invent one yourself).

4. (6×4 points) Find solutions, if they exist, for the following boundary value problems:
   a) $y'' + \pi^2 y = 0, \quad y(0) = 2, \quad y(1) = 0$,
   b) $y'' + \pi^2 y = 0, \quad y(0) = 1, \quad y(1) = 2$.
   c) $y'' + 9y = 0, \quad y(0) = 0, y(\pi) = 0$,
   d) $y'' + 9y = x, \quad y(0) = 0, y(\pi) = 0$.
   e) $y'' + 9y = \sin x, \quad y(0) = 0, y(\pi) = 1$,
   f) $y'' + 9y = \sin x, \quad y(0) = 1, y(\pi) = -1$.

5. (8 points) Discuss the consistency of your answers to Problem 4a-f with the two general theorems stated below [beware of unfortunate standard notation: homogenous below refers to the *boundary conditions*, not to a zero right-hand side of the ODE]:
I. An linear second-order ODE with nonhomogeneous boundary conditions (i.e. boundary conditions of the form $\alpha_i y(b_i) + \beta_i y'(b_i) = A_i$ at the two boundaries b_1 and b_2, with $A_1 \neq 0$ and/or $A_2 \neq 0$) has a unique solution if and only if the corresponding homogenous problem (with $A_1 = A_2 = 0$) has only the trivial solution $y = 0$.
II. If the homogenous boundary condition problem has a nontrivial solution, the corresponding nonhomogeneous boundary value problem either has no solution or an infinity of solutions given by $y = y_p(x) + cu(x)$ where y_p is a particular solution of the nonhomogeneous problem, c is an arbitrary constant and $u(x)$ is a nontrivial solution of the homogenous boundary problem.

6. (10 points) Let λ_i be the eigenvalues and y_i be the corresponding eigenfunctions of a linear differential operator L: $Ly_i + \lambda_i y_i = 0$. Show that a solution of

$$Ly + \lambda y = \sum_{k=1}^{n} A_k y_k \tag{8}$$

where the A_k are given constants is

$$y = \sum_{k=1}^{n} \frac{A_k y_k}{\lambda - \lambda_k} \tag{9}$$

provided λ is *not* an eigenvalue.

7. (10 points) Solve $y'' + \lambda y = \sin \pi x + 2 \sin 2\pi x + 3 \sin 3\pi x, \quad y(0) = y(1) = 0$, by using the method of problem 6. What if any restriction on λ is required so that this solution be valid?

Total points: 93