ACM95/100b
Problem Set 3 Solutions
01/27/04

# Problem 1 (4×5 points)

The following (trivial once you 'get it') problem is designed to help those of you who had trouble with Problem Set 1's problem 7c. It will also help you to find asymptotic expansions of Laplace integrals and their relatives.

a) Evaluate

$$\int_0^a \exp(-1000\,t)\,dt \tag{1}$$

for a= $10^{-4}$, $10^{-3}$, $3\times10^{-3}$, $9\times10^{-3}$, .1, 10 and ∞. Also give your answers in fixed decimal form to 7 digits [i.e. numbers like 0.1234567], and explain any trend you notice.

b) Now consider the integral

$$I(a) = \int_0^a \exp(-x\,t)\,dt \tag{2}$$

for x>>100. What range of values makes I(a)=I(∞)(1+ϵ) with |ϵ|<0.01 (i.e. gives I(∞) to 1% accuracy)? Your answer may depend on x.

c) Use reasoning motivated by part (b) to find the simplest function of x which approximates

$$I(x) = \int_0^\infty \frac{e^{-x\,t}}{1+t^2}\,dt \tag{3}$$

to 1% accuracy for x>>100. Justify your error estimate.

d) Use reasoning motivated by part (b) to find the simplest function of x which approximates

$$I(x) = \int_0^{1/5} \frac{e^{-xt}}{\sqrt{2\,t + t^3/4 + \cos t}}\,dt \tag{4}$$

to 1% accuracy for x>>100. Justify your error estimate.

# Solution to Problem 1

a)

$$J(a) = \int_0^a \exp(-1000\,t)\,dt = \frac{1-e^{-1000\,a}}{1000} \tag{5}$$

| a | J(a) |
|---|---|
| $10^{-4}$ | 0.0000952 |
| $10^{-3}$ | 0.0006321 |
| $3\times10^{-3}$ | 0.0009502 |
| $9\times10^{-3}$ | 0.0009999 |
| 0.1 | 0.0010000 |
| 10 | 0.0010000 |
| ∞ | 0.001 |

**Table 1**

For a>0.01 there is no difference between J(a) and J(∞) to seven decimal digits.

b)

$$I(a) = \int_0^a \exp(-x\,t)\,dt = \frac{1 - e^{-a\,x}}{x} \tag{6}$$

Since x>>100>0 we have a simple expression for I(∞)

$$I(\infty) = \int_0^\infty \exp(-x\,t)\,dt = \frac{1}{x} \tag{7}$$

This gives:

$$\left|\frac{I(a) - I(\infty)}{I(\infty)}\right| = e^{-a\,x} \tag{8}$$

We require this to be less than 0.01. This gives us an inequality involving x and a:

$$e^{-a\,x} < 0.01 \tag{9}$$

Since the exponential is a monotone function the inequality is preserved when we compute the natural log of both sides

$$-a\,x < \ln(0.01) \tag{10}$$

Further simplification gives:

$$a > -\ln(0.01)/x = 4.6051702/x \tag{11}$$

So for x>>100 we have found a suitable approximation provided that a>0.0461

c)

$$I(x) = \int_0^\infty \frac{e^{-x\,t}}{1 + t^2}\,dt \tag{12}$$

We want to find a function f(x) such that:

$$\left|\frac{I(x) - f(x)}{f(x)}\right| < 0.01 \tag{13}$$

Notice that for large values of x, the integrand is nearly zero except in a very small interval near t=0. This is made clear by the following plot of the integrand for x=100.

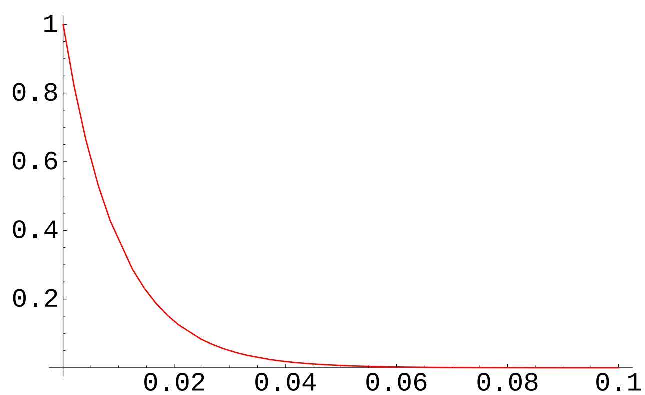


**Figure 1**

The smallness of the integrand away from t=0 seems to suggest that the integral might be approximated well by Taylor expanding $1/(1+t^2)$ near t=0 and keeping only the first term:

$$f(x) = \int_0^\infty e^{-xt}\, dt = \frac{1}{x} \qquad (14$$

The error in such an approximation is on the order of the integral of the next term in the Taylor series:

$$\text{Error}(x) = O\left(\int_0^\infty t^2 e^{-xt}\, dt\right) = O\left(\frac{2}{x^3}\right) \qquad (15)$$

This statement is made more precise by what's known as Watson's Lemma (see Bender & Orszag). For our purposes we'll take this to be a good measure of the error. We find:

$$\left|\frac{I(x) - f(x)}{f(x)}\right| \sim \left|\frac{2/x^3}{1/x}\right| = \frac{2}{x^2} << \frac{2}{100^2} = 0.0002 \qquad (16)$$

This is below the desired 1% threshold. For a more precise treatment, see the appendix. Below is a plot of the percent error as a function of x:

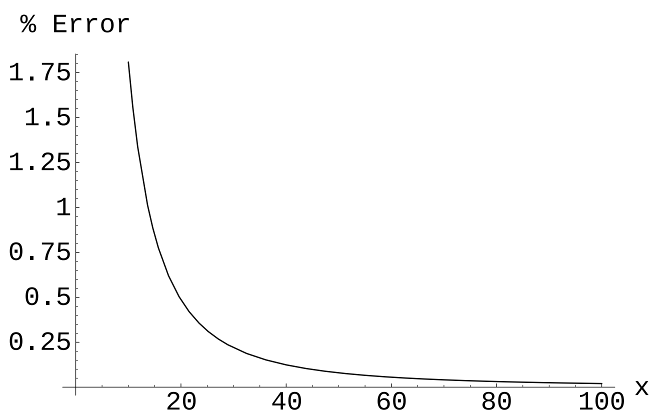


**Figure 2**

d)

$$I(x) = \int_0^{1/5} \frac{e^{-xt}}{\sqrt{2t + t^3/4 + \cos t}}\, dt \qquad (17)$$

Again the integrand is very small for t>0 as we see from this plot when x=100.

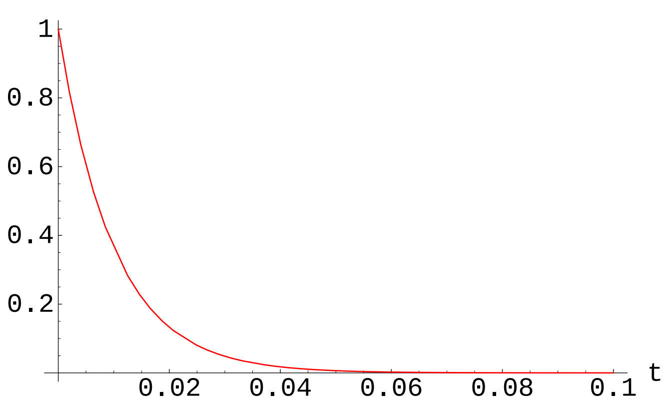


**Figure 3**

Because of this, only a small error is made by replacing I(x) with the integral over a semi-infinite interval:

$$J(x) = \int_0^\infty \frac{e^{-x\,t}}{\sqrt{2\,t + t^3/4 + \text{Cos}\,t}}\,dt \quad (18$$

The notion of "small error" is made precise by Watson's Lemma, however it is easy to see that

$$|J(x) - I(x)| = \int_{1/5}^\infty \frac{e^{-x\,t}}{\sqrt{2\,t + t^3/4 + \text{Cos}\,t}}\,dt \le \left(\frac{1}{\sqrt{2\,t + t^3/4 + \text{Cos}\,t}}\right)_{t=1/5} \int_{1/5}^\infty e^{-x\,t}\,dt =$$
$$0.85062\,\frac{e^{-x/5}}{x} \ll 1.75326 \times 10^{-11} \quad (19)$$

Now we need to approximate J(x). Proceeding as in part (c). We use the first term in the Taylor series:

$$\frac{1}{\sqrt{2\,t + t^3/4 + \text{Cos}\,t}} = 1 - t + \ldots \quad (20)$$

and we find:

$$f(x) = \int_0^\infty e^{-xt}\,dt = \frac{1}{x}$$
$$\text{Error}(x) = O\left(\int_0^\infty -t\,e^{-xt}\,dt\right) = \frac{-1}{x^2} \quad (21)$$

So we find:

$$\left|\frac{J(x) - f(x)}{f(x)}\right| \sim \left|\frac{1/x^2}{1/x}\right| = \frac{1}{x} \ll \frac{1}{100} = 0.01 \quad (22)$$

Combining this with the previous error bound and using the triangle inequality gives:

$$\left|\frac{I(x) - f(x)}{f(x)}\right| = \left|\frac{I(x) - J(x) + J(x) - f(x)}{f(x)}\right| \le$$
$$\left|\frac{I(x) - J(x)}{f(x)}\right| + \left|\frac{J(x) - f(x)}{f(x)}\right| \sim 0.85062\,\frac{e^{-x/5}}{x} + \frac{1}{x} \ll 1.75326 \times 10^{-11} + 0.01 \sim 0.01 \quad (23)$$

Therefor f(x)=1/x gives us an approximation to I(x) accurate to 1%. Below is a plot of % Error as a function of x.

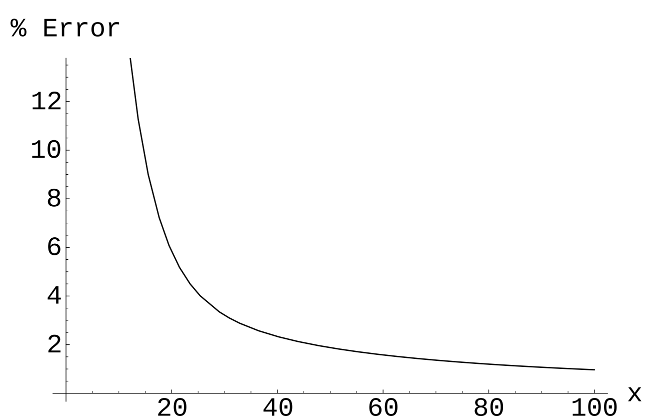


**Figure 4**

Comparing to the figure from part (c) shows that the approximation is not as good, because the Taylor series for the integrand in (c) had a second term that was $O(t^2)$ while the Taylor series for the integrand in (d) had a second term that was O(t).

# Problem 2 (5 points)

Prove that for an analytic function f(t) with Laplace transform F(s)

$$\lim_{s\to\infty} s\,F(s) = \lim_{t\to 0^+} f(t) \tag{24}$$

# Solution to Problem 2

$$F(s) = \int_0^\infty e^{-s\,t}\, f(t)\, dt \tag{25}$$

For very large positive Re(s), the integrand will be very small for all t>0. Motivated by the last problem, we suspect that a good approximation to this integral will come from using the first term in the Taylor series for f(t):

$$\int_0^\infty e^{-s\,t}\, f(0)\, dt = \frac{f(0)}{s} \tag{26}$$

To see if this is a good approximation we check the error:

$$\left|F(s) - \frac{f(0)}{s}\right| = \left|\int_0^\infty e^{-s\,t}\, f(t)\, dt - \int_0^\infty e^{-s\,t}\, f(0)\, dt\right| = \left|\int_0^\infty e^{-s\,t}\, (f(t) - f(0))\, dt\right| \le \int_0^\infty e^{-s\,t}\, |f(t) - f(0)|\, dt \tag{27}$$

Since f is analytic, it is differentiable, so, by the mean value theorem we have for some c∈[0,t]:

$$|f(t) - f(0)| = |t\, f'(c)| \tag{28}$$

So our bound becomes:

$$\left|F(s) - \frac{f(0)}{s}\right| \le |f'(c)| \int_0^\infty t\, e^{-s\,t}\, dt = \frac{|f'(c)|}{s^2} \tag{29}$$

Hence as s→∞ we have:

$$|s\,F(s) - f(0)| \le \frac{|f'(c)|}{s} \to 0 \tag{30}$$

So we have shown that:

$$\lim_{s\to\infty} s\,F(s) = \lim_{t\to 0^+} f(t) \tag{31}$$

# Problem 3 (6×4 points)

Fun with Dirac

a) Show that

$$x\,\frac{d}{dx}\,\delta(x) = -\delta(x) \tag{32}$$

(Hint: use the Gaussian $\delta$ sequence, $\delta_n$=n/$\sqrt{\pi}$ exp(-$n^2$ $x^2$); integration by parts may also be helpful)

b) Show that for f'(x) continuous at x=0

$$\int_{-\infty}^\infty \delta'(x)\, f(x)\, dx = -f'(0) \tag{33}$$

c) Show that if $x_0$ is the solution of f($x_0$)=0,

$$\delta(f(x)) = \left|\frac{d f(x)}{dx}\right|^{-1} \delta(x - x_0) \tag{34}$$

d)What happens in (c) if f(x) has more than one zero? Use your result to find for $x_1 \neq x_2$

$$\delta[(x - x_1)(x - x_2)] \tag{35}$$

e) Consider the three dimensional delta function in cartesian coordinates:

$$\delta(r - r_0) \equiv \delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0) \tag{36}$$

Introduce a new coordinate system with the three new coordinates $\xi$, $\eta$, and $\zeta$ by x=X($\xi,\eta,\zeta$), y=Y($\xi,\eta,\zeta$), z=Z($\xi,\eta,\zeta$). Show that the equation of part (c) generalizes to

$$\delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0) = \delta(\xi - \xi_0)\,\delta(\eta - \eta_0)\,\delta(\zeta - \zeta_0)\left|\frac{\partial(X, Y, Z)}{\partial(\xi, \eta, \zeta)}\right|^{-1} \tag{37}$$

where ∂(X,Y,Z)/∂($\xi,\eta,\zeta$) is the Jacobian determinant of partial derivatives of X,Y,Z with respect to $\xi,\eta,\zeta$ and $\xi_0$, $\eta_0$, and $\zeta_0$ are the solutions of $x_0$= X( $\xi_0$, $\eta_0$, $\zeta_0$ ), $y_0$=Y($\xi_0$, $\eta_0$, $\zeta_0$), $z_0$=Z($\xi_0$, $\eta_0$, $\zeta_0$)

f) In particular, show that in cylindrical coordinates r, $\phi$, z:

$$\delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0) = \frac{1}{r}\,\delta(r - r_0)\,\delta(\phi - \phi_0)\,\delta(z - z_0) \tag{38}$$

in spherical coordinates r, $\theta$, $\phi$

$$\delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0) = \frac{1}{r^2\,\mathrm{Sin}\,\theta}\,\delta(r - r_0)\,\delta(\theta - \theta_0)\,\delta(\phi - \phi_0) \tag{39}$$

and if $\mu$= Cos $\theta$ is substituted in spherical coordinates

$$\delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0) = \frac{1}{r^2}\,\delta(r - r_0)\,\delta(\mu - \mu_0)\,\delta(\phi - \phi_0) \tag{40}$$

# Solution to Problem 3

a)

$$x\,\frac{d}{dx}\,\delta(x) = -\delta(x) \tag{41}$$

The delta function is defined by it's action on smooth test functions, i.e. the delta function is a "function" which satisfies

$$f(0) = \int_{-\infty}^{\infty}\delta(x)\,f(x)\,dx \tag{42}$$

for all infinitely differentiable (smooth) functions f(x) which are zero outside of some interval (-a,a) (i.e. have compact support). Let us consider

$$\int_{-\infty}^{\infty} x\,\delta_n{}'(x)\,f(x)\,dx \tag{43}$$

Method 1: Integrate by parts

$$\begin{aligned}\int_{-\infty}^{\infty} x\,\delta_n{}'(x)\,f(x)\,dx &= \lim_{x\to\infty}(x\,f(x)\,\delta_n(x)) - \lim_{x\to-\infty}(x\,f(x)\,\delta_n(x)) - \int_{-\infty}^{\infty}(f(x) + x\,f'(x))\,\delta_n(x)\,dx = \\ &- \int_{-\infty}^{\infty} f(x)\,\delta_n(x)\,dx - \int_{-\infty}^{\infty} x\,f'(x)\,\delta_n(x)\,dx\end{aligned} \tag{44}$$

Since the Gaussian delta sequence has $\delta$ as it's limit, we have:

$$\lim_{n\to\infty}\left(-\int_{-\infty}^{\infty} f(x)\,\delta_n(x)\,dx\right) = -f(0)$$
$$\lim_{n\to\infty}\left(-\int_{-\infty}^{\infty} x\,f'(x)\,\delta_n(x)\,dx\right) = -0\,f'(0) = 0 \tag{45}$$

So we conclude:

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} x\,\delta_n'(x)\,f(x)\,dx = -f(0) = \lim_{n\to\infty}\int_{-\infty}^{\infty} -\delta_n(x)\,f(x)\,dx \tag{46}$$

Hence $x\,\delta_n'(x)$ and $-\delta_n(x)$ have the same limit so that we must have:

$$x\,\delta'(x) = -\delta(x) \tag{47}$$

Method 2: Approximate the integrals directly

$$\int_{-\infty}^{\infty} x\,\delta_n'(x)\,f(x)\,dx = \int_{-\infty}^{\infty} x\left(\frac{n}{\sqrt{\pi}}\,e^{-n^2 x^2}\right)' f(x)\,dx = \frac{-2\,n^3}{\sqrt{\pi}}\int_{-\infty}^{\infty} x^2\,f(x)\,e^{-n^2 x^2}\,dx \tag{48}$$

By the same reasoning as in problem 1, we replace terms in the integrand with their taylor series expansion:

$$\frac{-2\,n^3}{\sqrt{\pi}}\int_{-\infty}^{\infty} x^2\,f(x)\,e^{-n^2 x^2}\,dx \sim \frac{-2\,n^3}{\sqrt{\pi}}\int_{-\infty}^{\infty} x^2\,f(0)\,e^{-n^2 x^2}\,dx = -f(0) \tag{49}$$

The error is given approximately by:

$$\text{Error} = O\left(\frac{-2\,n^3}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{1}{2}\,x^4\,f''(0)\,e^{-n^2 x^2}\,dx\right) = O\left(-\frac{3}{4\,n^2}\,f''(0)\right) \tag{50}$$

And this vanishes as n→∞. Likewise we find:

$$\int_{-\infty}^{\infty} -\delta_n(x)\,f(x)\,dx = -\frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-n^2 x^2}\,dx \sim -\frac{n\,f(0)}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-n^2 x^2}\,dx = -f(0) \tag{51}$$

So we conclude that:

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} x\,\delta_n'(x)\,f(x)\,dx = -f(0) = \lim_{n\to\infty}\int_{-\infty}^{\infty} -\delta_n(x)\,f(x)\,dx \tag{52}$$

and hence $x\,\delta'(x)=-\delta(x)$.

b)

$$\int_{-\infty}^{\infty}\delta'(x)\,f(x)\,dx = \lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n'(x)\,f(x)\,dx \tag{53}$$

Method 1: Integrate by parts

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n'(x)\,f(x)\,dx =$$
$$\lim_{n\to\infty}\left(\lim_{x\to\infty}(f(x)\,\delta_n(x)) - \lim_{x\to-\infty}(f(x)\,\delta_n(x)) - \int_{-\infty}^{\infty}\delta_n(x)\,f'(x)\,dx\right) = -\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(x)\,f'(x)\,dx = -f'(0) \tag{54}$$

We conclude:

$$\int_{-\infty}^{\infty}\delta'(x)\,f(x)\,dx = \lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n'(x)\,f(x)\,dx = -f'(0) \tag{55}$$

Method 2: Approximate the integral directly

Use the defiition of the delta sequence and keep two terms in the Taylor series:

$$\int_{-\infty}^{\infty} \delta_n'(x)\, f(x)\, dx = \frac{-2\, n^3}{\sqrt{\pi}} \int_{-\infty}^{\infty} x\, f(x)\, e^{-n^2 x^2}\, dx \sim \frac{-2\, n^3}{\sqrt{\pi}} \int_{-\infty}^{\infty} (x\, f(0) + x^2\, f'(0))\, e^{-n^2 x^2}\, dx = -f'(0) \quad (56)$$

The error is given by:

$$\text{Error} = O\left(\frac{-2\, n^3}{\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{6}\, x^4\, f'''(0)\right) e^{-n^2 x^2}\, dx\right) = O\left(\frac{-f'''(0)}{4\, n^2}\right) \quad (57)$$

This vanishes as $n \to \infty$. We conclude:

$$\int_{-\infty}^{\infty} \delta'(x)\, f(x)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n'(x)\, f(x)\, dx = -f'(0) \quad (58)$$

c)

Let g(x) be a test function and suppose that f(x) has only one zero, located at $x_0$. Furthermore, suppose that the function f(x) is invertible near $x_0$, i.e. by the inverse function theorem we require $f'(x_0) \neq 0$. We wish to calculate:

$$\int_{-\infty}^{\infty} \delta(f(x))\, g(x)\, dx \quad (59)$$

Since $\delta(x)$ is zero outside of any neighborhood of $x_0$, we could replace the integral with one about $x_0$ where the function f(x) has an inverse. Since $f'(x_0) \neq 0$ the inverse of f exists in some neighborhood of $x_0$ which we'll call $(x_0-a, x_0+b)$. These two observations let us write:

$$\boxed{\int_{-\infty}^{\infty} \delta(f(x))\, g(x)\, dx = \int_{x_0-a}^{x_0+b} \delta(f(x))\, g(x)\, dx} \quad (60)$$

and then change variables according to y=f(x):

$$\int_{f(x_0-a)}^{f(x_0+b)} \delta(y)\, \frac{g(f^{-1}(y))}{f'(f^{-1}(y))}\, dy \quad (61)$$

Since $f' \neq 0$ in this interval, it is always of one sign. If it is positive, then the range of integration is positively oriented, and by the definition of the delta function, this last integral is:

$$\frac{g(f^{-1}(0))}{f'(f^{-1}(0))} \quad (62)$$

If $f'<0$ then the range of integration is negatively oriented and the integral is

$$-\frac{g(f^{-1}(0))}{f'(f^{-1}(0))} \quad (63)$$

So in general, this integral is

$$\frac{g(f^{-1}(0))}{|f'(f^{-1}(0))|} \quad (64)$$

Recalling that $f(x_0)=0$ gives:

$$\boxed{\int_{-\infty}^{\infty} \delta(f(x))\, g(x)\, dx = \frac{g(f^{-1}(0))}{|f'(f^{-1}(0))|} = \left|\left(\frac{df}{dx}\right)_{x=x_0}\right|^{-1} g(x_0)} \quad (65)$$

Also notice that:

$$\int_{-\infty}^{\infty} \delta(x - x_0) \left| \frac{d f}{d x} \right|^{-1} g(x)\, dx = \left| \left( \frac{d f}{d x} \right)_{x = x_0} \right|^{-1} g(x_0) \tag{66}$$

So we have shown that when f is a differentiable function with exactly one zero at $x_0$ with f'≠0 then:

$$\delta(f(x)) = \left| \frac{d f}{d x} \right|^{-1} \delta(x - x_0) \tag{67}$$

d)

Suppose f has countably many zeros at the points $x_i$. Then since $\delta$(x)=0 for x≠0 we may write the integral

$$\int_{-\infty}^{\infty} \delta(f(x))\, g(x)\, dx \tag{68}$$

as countably many integrals about each of the $x_i$. Further, we require that f'($x_i$)≠0 so that f is locally invertible near each of the $x_i$. Let the neighborhood where the local inverse is valid be $(x_i - a_i, x_i + b_i)$. Putting these facts together gives:

$$\int_{-\infty}^{\infty} \delta(f(x))\, g(x)\, dx = \sum_{i=1}^{\infty} \int_{x_i - a_i}^{x_i + b_i} \delta(f(x))\, g(x)\, dx \tag{69}$$

and allows us to evaluate each of the integrals by the same method as the previous problem. We find:

$$\int_{-\infty}^{\infty} \delta(f(x))\, g(x)\, dx = \sum_{i=1}^{\infty} \left| \left( \frac{d f}{d x} \right)_{x = x_i} \right|^{-1} g(x_i) \tag{70}$$

This shows:

$$\boxed{\delta(f(x)) = \sum_{i=1}^{\infty} \left| \frac{d f}{d x} \right|^{-1} \delta(x - x_i)} \tag{71}$$

As a special case, consider the function:

$$f(x) = (x - x_1)(x - x_2) \tag{72}$$

Using the formula just derived we have:

$$\boxed{\delta((x - x_1)(x - x_2)) = \sum_{i=1}^{2} \left| \left( \frac{d}{d x} (x - x_1)(x - x_2) \right)_{x = x_i} \right|^{-1} \delta(x - x_i) = \frac{\delta(x - x_1) + \delta(x - x_2)}{|x_1 - x_2|}} \tag{73}$$

e)

In part (c) we showed that for an invertible 1-D function f(y) with a single zero at $y_0$ the following is true:

$$\delta(f(y)) = \left| \frac{d f}{d y} \right|^{-1} \delta(y - y_0) \tag{74}$$

We now prove a similar result for higher dimensions. In what follows x's and y's are vectors, and f is a vector valued function. Let us calculate:

$$\int \delta(f(y))\, g(y)\, dy \tag{75}$$

where the integral is taken over all of n-dimensional space. Suppose that f is invertible (i.e. has non-zero jacobian) at the point $y_0$ and that $y_0$ is the only zero of f(y). Since $\delta(x)=0$ for $x\neq 0$ this integral may be confined to the region around x=0 where f is invertible. Call this region R.

$$\int \delta(f(y))\, g(y)\, dy = \int_R \delta(f(y))\, g(y)\, dy \tag{76}$$

If under the inverse mapping, R is mapped to S, then the formula for this new integral comes from any advanced calculus text (e.g. Apostol, vol2). Recall from calculus, that under the mapping x=h(u) the following formula holds

$$\int k(x)\, dx = \int k(h(u))\, J(u)\, du \tag{77}$$

Where J(u) is the jacobian of the function h(u). Letting y= $f^{-1}(z)$ This formula gives:

$$\int_R \delta(f(y))\, g(y)\, dy = \int_S \delta(z)\, g(f^{-1}(z))\, J(z)\, dz \tag{78}$$

Where J(z) is the Jacobian of the inverse function $f^{-1}(z)$. From the definition of the delta function, this last integral is

$$g(f^{-1}(0))\, J(0) = g(y_0)\, J(0) = g(y_0) \left| \left( \frac{\partial f^{-1}}{\partial z} \right)_{z=0} \right| \tag{79}$$

Also from calculus, the Jacobian of $f^{-1}(z)$ is 1 divided by the Jacobian of f(y). So we have:

$$\int \delta(f(y))\, g(y)\, dy = g(y_0) \left| \left( \frac{\partial f}{\partial y} \right)_{y=y_0} \right|^{-1} \tag{80}$$

We have thus shown that:

$$\delta(f(y)) = \delta(y - y_0) \left| \frac{\partial f}{\partial y} \right|^{-1} \tag{81}$$

If the function f is that which maps y→x and $y_0 \to x_0$, then we may write:

$$\delta(x - x_0) = \delta(y - y_0) \left| \frac{\partial f}{\partial y} \right|^{-1} \tag{82}$$

In coordinate form in 3-D this is:

$$\delta(x_1 - x_{10})\, \delta(x_2 - x_{20})\, \delta(x_3 - x_{30}) = \delta(y_1 - y_{10})\, \delta(y_2 - y_{20})\, \delta(y_3 - y_{30}) \left| \frac{\partial(f_1, f_2, f_3)}{\partial(y_1, y_2, y_3)} \right|^{-1} \tag{83}$$

A quick change to Greek variables gives the desired answer.

f)

If coordinates are being changed according to

$$\begin{aligned} x &= X(\xi, \eta, \zeta) \\ y &= Y(\xi, \eta, \zeta) \\ z &= Z(\xi, \eta, \zeta) \end{aligned} \tag{84}$$

then the Jacobian appearing above is:

$$\begin{vmatrix} \frac{\partial X}{\partial \xi} & \frac{\partial Y}{\partial \xi} & \frac{\partial Z}{\partial \xi} \\ \frac{\partial X}{\partial \eta} & \frac{\partial Y}{\partial \eta} & \frac{\partial Z}{\partial \eta} \\ \frac{\partial X}{\partial \zeta} & \frac{\partial Y}{\partial \zeta} & \frac{\partial Z}{\partial \zeta} \end{vmatrix} \quad (85$$

For cylindrical coordinates the change is given by:

$$\begin{aligned} x &= X(r, \phi, z) = r \operatorname{Cos} \phi \\ y &= Y(r, \phi, z) = r \operatorname{Sin} \phi \\ z &= Z(r, \phi, z) = z \end{aligned} \quad (86)$$

So the jacobian is

$$\begin{vmatrix} \frac{\partial X}{\partial r} & \frac{\partial Y}{\partial r} & \frac{\partial Z}{\partial r} \\ \frac{\partial X}{\partial \phi} & \frac{\partial Y}{\partial \phi} & \frac{\partial Z}{\partial \phi} \\ \frac{\partial X}{\partial z} & \frac{\partial Y}{\partial z} & \frac{\partial Z}{\partial z} \end{vmatrix} = \begin{vmatrix} \operatorname{Cos} \phi & \operatorname{Sin} \phi & 0 \\ -r \operatorname{Sin} \phi & r \operatorname{Cos} \phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \operatorname{Cos} \phi & \operatorname{Sin} \phi \\ -r \operatorname{Sin} \phi & r \operatorname{Cos} \phi \end{vmatrix} = r \quad (87)$$

The formula from part (e) then gives the desired answer.

$$\delta(x - x_0)\, \delta(y - y_0)\, \delta(z - z_0) = \frac{1}{r}\, \delta(r - r_0)\, \delta(\phi - \phi_0)\, \delta(z - z_0) \quad (88)$$

in spherical coordinates r, $\theta$, $\phi$ the change is given by

$$\begin{aligned} x &= X(r, \theta, \phi) = r \operatorname{Cos} \phi \operatorname{Sin} \theta \\ y &= Y(r, \theta, \phi) = r \operatorname{Sin} \phi \operatorname{Sin} \theta \\ z &= Z(r, \theta, \phi) = r \operatorname{Cos} \theta \end{aligned} \quad (89)$$

So the jacobian is

$$\begin{vmatrix} \frac{\partial X}{\partial r} & \frac{\partial Y}{\partial r} & \frac{\partial Z}{\partial r} \\ \frac{\partial X}{\partial \theta} & \frac{\partial Y}{\partial \theta} & \frac{\partial Z}{\partial \theta} \\ \frac{\partial X}{\partial \phi} & \frac{\partial Y}{\partial \phi} & \frac{\partial Z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \operatorname{Cos} \phi \operatorname{Sin} \theta & \operatorname{Sin} \theta \operatorname{Sin} \phi & \operatorname{Cos} \theta \\ r \operatorname{Cos} \phi \operatorname{Cos} \theta & r \operatorname{Sin} \phi \operatorname{Cos} \theta & -r \operatorname{Sin} \theta \\ -r \operatorname{Sin} \phi \operatorname{Sin} \theta & r \operatorname{Sin} \theta \operatorname{Cos} \phi & 0 \end{vmatrix} = r^2 \operatorname{Sin} \theta \quad (90)$$

$$\delta(x - x_0)\, \delta(y - y_0)\, \delta(z - z_0) = \frac{1}{r^2 \operatorname{Sin} \theta}\, \delta(r - r_0)\, \delta(\theta - \theta_0)\, \delta(\phi - \phi_0) \quad (91)$$

and if $\mu$= Cos $\theta$ is substituted in spherical coordinates

$$\begin{aligned} x &= X(r, \mu, \phi) = r \operatorname{Cos} \phi\, (1 - \mu^2)^{1/2} \\ y &= Y(r, \mu, \phi) = r \operatorname{Sin} \phi\, (1 - \mu^2)^{1/2} \\ z &= Z(r, \mu, \phi) = r\, \mu \end{aligned} \quad (92)$$

$$\begin{vmatrix} \frac{\partial X}{\partial r} & \frac{\partial Y}{\partial r} & \frac{\partial Z}{\partial r} \\ \frac{\partial X}{\partial \mu} & \frac{\partial Y}{\partial \mu} & \frac{\partial Z}{\partial \mu} \\ \frac{\partial X}{\partial \phi} & \frac{\partial Y}{\partial \phi} & \frac{\partial Z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \operatorname{Cos} \phi\, (1 - \mu^2)^{1/2} & \operatorname{Sin} \phi\, (1 - \mu^2)^{1/2} & \mu \\ -\mu r \operatorname{Cos} \phi\, (1 - \mu^2)^{-1/2} & -r\, \mu\, (1 - \mu^2)^{-1/2} \operatorname{Sin} \phi & 0 \\ -r \operatorname{Sin} \phi\, (1 - \mu^2)^{1/2} & r\, (1 - \mu^2)^{1/2} \operatorname{Cos} \phi & r \end{vmatrix} = r^2 \quad (93)$$

Applying the formula from part (e) gives:

$$\delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0) = \frac{1}{r^2}\,\delta(r - r_0)\,\delta(\mu - \mu_0)\,\delta(\phi - \phi_0) \quad (94$$

# Problem 4 (10 points)

Laplace transforms on discontinuous functions

a) (4 points) Consider the function

$$\phi(t) = \begin{matrix} 0 & t < t_0 \\ 1 & t > t_0 \\ \phi_0 & t = t_0 \end{matrix} \tag{95}$$

Compute the Laplace transform Φ(s) of $\phi$(t). Then by explicit integration of the Mellin inversion formula along the Bromwich contour (being careful about principle value when needed), show that the inverse Laplace transform of Φ(s) is

$$\{\mathcal{L}^{-1}\,\Phi\}(t) = \begin{matrix} 0 & t < t_0 \\ 1 & t > t_0 \\ 1/2 & t = t_0 \end{matrix} \tag{96}$$

and thus need not agree with $\phi$(t) at t=$t_0$.

b) (2 points) Is this consistent with Lerch's theorem (as stated in class: if $f_1(t)$ and $f_2(t)$ have the same Laplace transform, then $f_1$ and $f_2$ differ by a null function, i.e. $f_1 - f_2 = N(t)$, where $\int_0^{t_0} N(t)\,dt$=0 for all $t_0$>0.)?

c) (4 points) Now consider

$$\psi(t) = \begin{matrix} f(t) & t < t_0 \\ g(t) & t > t_0 \end{matrix} \tag{97}$$

By considering the continuous (except possibly at the isolated point $t_0$) function

$$\psi(t) - H(t - t_0)\,(g(t_0^{+}) - f(t_0^{-})) \tag{98}$$

find $[\mathcal{L}^{-1}\,\{\mathcal{L}(\psi)\}](t_0)$?

# Solution to Problem 4

$$\phi(t) = \begin{matrix} 0 & t < t_0 \\ 1 & t > t_0 \\ \phi_0 & t = t_0 \end{matrix} \tag{99}$$

Compute the transform from the definition:

$$\Phi(s) = \int_0^{\infty} e^{-s\,t}\,\phi(t)\,dt = \int_0^{t_0} e^{-s\,t}\,0\,dt + \int_{t_0}^{\infty} e^{-s\,t}\,1\,dt = \frac{e^{-t_0\,s}}{s} \tag{100}$$

Compute the inverse from the inversion formula:

$$\{\mathcal{L}^{-1}\,\Phi\}(t) = \frac{1}{2\,\pi\,i}\int_{c-i\infty}^{c+i\infty} e^{st}\,\frac{e^{-t_0\,s}}{s}\,ds = \frac{1}{2\,\pi\,i}\int_{c-i\infty}^{c+i\infty} \frac{e^{(t-t_0)\,s}}{s}\,ds \tag{101}$$

case (i): $t > t_0$

Let c be any real number and integrate along the following contour:

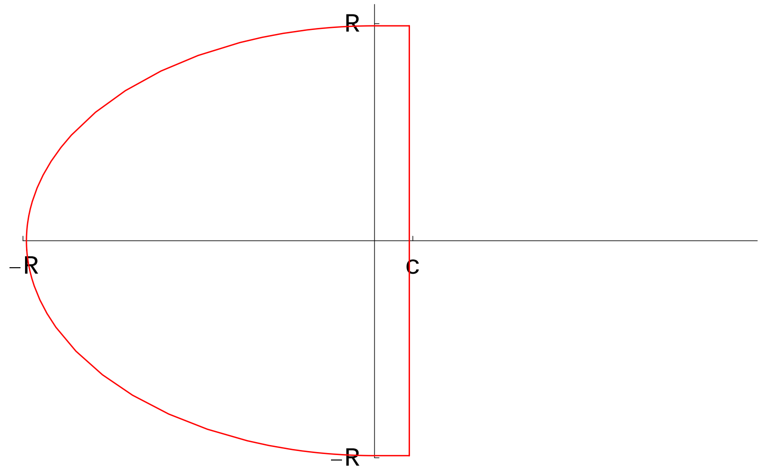


**Figure 5**

Call this contour $\Gamma_R$ and call the semicircular part $C_R$. By the residue theorem we have:

$$2\pi i = 2\pi i \, \text{Res}\left(\frac{e^{(t-t_0)z}}{z}, 0\right) = \int_{\Gamma_R} \frac{e^{(t-t_0)z}}{z} \, dz \tag{102}$$

Integrating counterclockwise along each piece of the contour separately gives:

$$\int_{\Gamma_R} \frac{e^{(t-t_0)z}}{z} \, dz = \int_{c-iR}^{c+iR} \frac{e^{(t-t_0)z}}{z} \, dz + \int_{c+iR}^{iR} \frac{e^{(t-t_0)z}}{z} \, dz + \int_{C_R} \frac{e^{(t-t_0)z}}{z} \, dz + \int_{-iR}^{c-iR} \frac{e^{(t-t_0)z}}{z} \, dz \tag{103}$$

Each of these integrals may be computed or approximated:

$$\begin{gathered} \left| \int_{c+iR}^{iR} \frac{e^{(t-t_0)z}}{z} \, dz \right| = \left| e^{(t-t_0)\, iR} \int_0^c \frac{e^{(t-t_0)y}}{y + iR} \, dy \right| = \\ \left| \int_0^c \frac{e^{(t-t_0)y}}{y + iR} \, dy \right| \le \int_0^c \left| \frac{e^{(t-t_0)y}}{y + iR} \right| dy = \int_0^c \frac{e^{(t-t_0)y}}{\sqrt{y^2 + R^2}} \, dy \le c \, \frac{e^{(t-t_0)c}}{R} \to 0 \end{gathered} \tag{104}$$

Similarly

$$\left| \int_{c-iR}^{-iR} \frac{e^{(t-t_0)z}}{z} \, dz \right| \to 0 \tag{105}$$

Now examine the semicircular part:

$$\left| \int_{C_R} \frac{e^{(t-t_0)z}}{z} \, dz \right| = \left| \int_{\pi/2}^{3\pi/2} \frac{e^{(t-t_0)R e^{i\theta}}}{R e^{i\theta}} \, iR e^{i\theta} \, d\theta \right| \le \int_{\pi/2}^{3\pi/2} \left| e^{(t-t_0)R e^{i\theta}} \right| d\theta = \int_{\pi/2}^{3\pi/2} e^{(t-t_0)R \cos\theta} \, d\theta \tag{106}$$

Since $\cos\theta$ is non-positive in this region, and $t - t_0 > 0$, this integral vanishes by Jordan's Lemma.

So we have shown for $t > t_0$:

$$2\pi i = \lim_{R\to\infty} \int_{c-iR}^{c+iR} \frac{e^{(t-t_0)z}}{z} \, dz \tag{107}$$

So for $t > t_0$ we have

$$\{\mathcal{L}^{-1}\,\Phi\}(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{(t-t_0)s}}{s} \, ds = 1 \tag{108}$$

case (ii) $t < t_0$

Now use the contour drawn below:

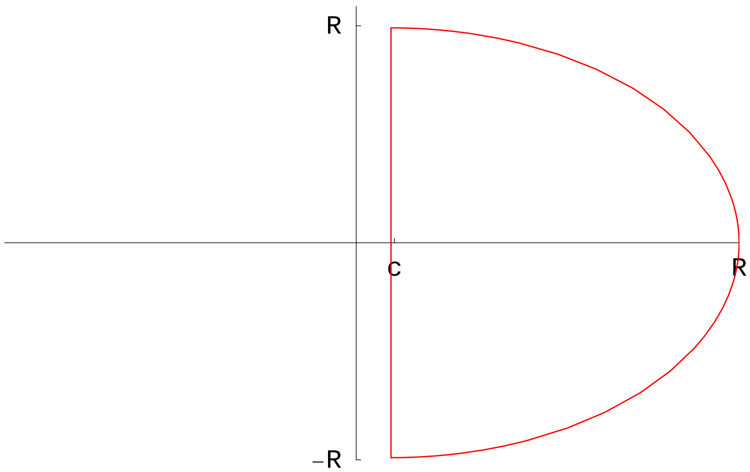


**Figure 6**

By the Cauchy-Goursat theorem, the integral around this contour is zero.

$$0 = \int_{\Gamma_R} \frac{e^{(t-t_0)z}}{z}\, dz = \int_{c-iR}^{c+iR} \frac{e^{(t-t_0)z}}{z}\, dz + \int_{C_R} \frac{e^{(t-t_0)z}}{z}\, dz \qquad (109)$$

Bounding the semicircular contour:

$$\left| \int_{C_R} \frac{e^{(t-t_0)z}}{z}\, dz \right| = \left| \int_{\pi/2}^{-\pi/2} \frac{e^{(t-t_0)(c+R\,e^{i\theta})}}{c+R\,e^{i\theta}}\, i\,R\,e^{i\theta}\, d\theta \right| \leq$$
$$R\,e^{(t-t_0)c} \int_{\pi/2}^{-\pi/2} \left| \frac{e^{(t-t_0)R\,e^{i\theta}}}{c+R\,e^{i\theta}} \right| d\theta \leq \frac{R\,e^{(t-t_0)c}}{R-c} \int_{\pi/2}^{-\pi/2} e^{(t-t_0)R\,\mathrm{Cos}\,\theta}\, d\theta \qquad (110)$$

Since Cos $\theta$ is positive in this region and $t - t_0 < 0$ Jordan's lemma tells us that this integral vanishes.

So we have shown for $t < t_0$:

$$\{\mathcal{L}^{-1}\,\Phi\}(t) = \frac{1}{2\pi i} \int_{c-iR}^{c+iR} \frac{e^{(t-t_0)z}}{z}\, dz = 0 \qquad (111)$$

case (iii) $t = t_0$

$$\{\mathcal{L}^{-1}\,\Phi\}(t_0) = \frac{1}{2\pi i} \int_{c-iR}^{c+iR} \frac{1}{z}\, dz \qquad (112)$$

The integrand is analytic along the contour provided c>0. So we may evaluate this integral using anti-derivatives provided that we define a branch for the log whose cut doesn't cross the contour. We define our log by

$$-\pi < \arg(z) < \pi \qquad (113)$$

So we have:

$$\int_{c-iR}^{c+iR} \frac{1}{z}\, dz = \mathrm{Log}(c+i\,R) - \mathrm{Log}(c-i\,R) = \ln\left|\frac{c+i\,R}{c-i\,R}\right| + i\,(\arg(c+i\,R) - \arg(c-i\,R)) \qquad (114)$$

Obviously

$$\ln\left|\frac{c+iR}{c-iR}\right| \to \ln|-1| = 0 \tag{115}$$

Also, for our branch, we have:

$$\begin{aligned} &\arg(c+iR) \to \pi/2 \\ &\arg(c-iR) \to -\pi/2 \end{aligned} \tag{116}$$

So that we have:

$$\{\mathcal{L}^{-1}\,\Phi\}(t_0) = \frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{1}{z}\,dz = \frac{1}{2\pi i}\,i\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = \frac{1}{2} \tag{117}$$

b)

Define the following two functions:

$$f_1(t) = \begin{matrix} 0 & t < t_0 \\ 1 & t > t_0 \\ \phi_0 & t = t_0 \end{matrix} \tag{118}$$

$$f_2(t) = \begin{matrix} 0 & t < t_0 \\ 1 & t > t_0 \\ 1/2 & t = t_0 \end{matrix} \tag{119}$$

Their difference is:

$$N(t) = \begin{matrix} 0 & t \neq t_0 \\ \phi_0 - 1/2 & t = t_0 \end{matrix} \tag{120}$$

This function N(t) is indeed a null function since:

$$\int_0^t N(s)\,ds = 0 \quad \text{for all s} \tag{121}$$

So even though $f_1 \neq f_2$ pointwise, the integral of their difference is always zero since the functions differ only on a set of measure 0 (in this case, a point). So our previous results are consistent with Lerch's theorem.

c)

$$\psi(t) = \begin{matrix} f(t) & t < t_0 \\ g(t) & t > t_0 \end{matrix} \tag{122}$$

$$h(t) = \psi(t) - H(t - t_0)\,(g(t_0{}^+) - f(t_0{}^-)) \tag{123}$$

Notice that:

$$\begin{aligned} &\lim_{t\to t_0{}^-} h(t) = f(t_0{}^-) \\ &\lim_{t\to t_0{}^+} h(t) = f(t_0{}^-) \end{aligned} \tag{124}$$

So that h(t) is a continuous function. Transforming h(t) gives:

$$\mathcal{L}(h) = \mathcal{L}(\psi) - (g(t_0{}^+) - f(t_0{}^-))\int_{t_0}^{\infty} e^{-s\,t}\,dt = \mathcal{L}(\psi) - (g(t_0{}^+) - f(t_0{}^-))\,\frac{e^{-t_0\,s}}{s} \tag{125}$$

Since h is continuous, we may invert both sides to get:

$$h(t) = \mathcal{L}^{-1}[\mathcal{L}(\psi)] - (g(t_0{}^+) - f(t_0{}^-))\,\mathcal{L}^{-1}\left[\frac{e^{-t_0 s}}{s}\right] \tag{126}$$

In part (a) we computed the inverse transform on the right:

$$\mathcal{L}^{-1}\left[\frac{e^{-t_0 s}}{s}\right] = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \\ 1/2 & t = t_0 \end{cases} \tag{127}$$

Thus we have:

$$\begin{aligned} &\mathcal{L}^{-1}[\mathcal{L}(\psi)](t) = h(t) = f(t) \quad \text{for } t < t_0 \\ &\mathcal{L}^{-1}[\mathcal{L}(\psi)](t_0) = h(t_0) + \frac{g(t_0{}^+) - f(t_0{}^-)}{2} = \frac{g(t_0{}^+) + f(t_0{}^-)}{2} \\ &\mathcal{L}^{-1}[\mathcal{L}(\psi)](t) = h(t) + g(t_0{}^+) - f(t_0{}^-) = g(t) \quad \text{for } t > t_0 \end{aligned} \tag{128}$$

# Problem 5 (2×7 points)

Use the shifting theorems to assist you in solving the following initial value problems (you may in addition use Laplace transform tables -e.g. the one handed out in class):

a)

$$\begin{aligned} &4y'' - 4y' + 37y = 0 \\ &y(0) = 3 \\ &y'(0) = 3/2 \end{aligned} \tag{129}$$

b)

$$\begin{aligned} &y'' + y = r(t) = \begin{cases} t & 0 < t < 1 \\ 0 & t > 1 \end{cases} \\ &y(0) = 0 \\ &y'(0) = 0 \end{aligned} \tag{130}$$

# Solution to Problem 5

a)

$$\begin{aligned} &4y'' - 4y' + 37y = 0 \\ &y(0) = 3 \\ &y'(0) = 3/2 \end{aligned} \tag{131}$$

Applying the Laplace transform gives:

$$4(s^2 Y - s\,y(0) - y'(0)) - 4(s\,Y - y(0)) + 37\,Y = 0 \tag{132}$$

Simplifying:

$$Y = \frac{12s - 6}{4s^2 - 4s + 37} = 3\,\frac{s - \frac{1}{2}}{(s - \frac{1}{2})^2 + 3^2} \tag{133}$$

Knowing one transform and a shifting theorem

$$\mathcal{L}(\text{Cos}\,k\,x) = \frac{s}{s^2 + k^2}$$
$$\{\mathcal{L}(e^{a\,x}\,f(x))\}(s) = \{\mathcal{L}(f(x))\}(s - a) \quad (134)$$

allows us to write:

$$y(x) = 3\,e^{\frac{x}{2}}\,\text{Cos}\,3\,x \quad (135)$$

b)

$$y'' + y = r(t) = \begin{matrix} t & 0 < t < 1 \\ 0 & t > 1 \end{matrix}$$
$$y(0) = 0$$
$$y'(0) = 0 \quad (136)$$

Applying the Laplace transform gives:

$$s^2\,Y - s\,y(0) - y'(0) + Y = \int_0^1 t\,e^{-s\,t}\,dt = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) \quad (137)$$

Simplifying:

$$Y = \frac{1}{s^2\,(s^2 + 1)} - \frac{e^{-s}}{s^2 + 1}\left(\frac{1}{s^2} + \frac{1}{s}\right) \quad (138)$$

We use partial fractions to rewrite this:

$$Y = \frac{1}{s^2} - \frac{1}{s^2 + 1} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s} - \frac{1}{s^2 + 1} - \frac{s}{s^2 + 1}\right) \quad (139)$$

We then use the following 4 transforms and a shifting theorem:

$$\mathcal{L}(\text{Sin}\,x) = \frac{1}{s^2 + 1}$$
$$\mathcal{L}(\text{Cos}\,x) = \frac{s}{s^2 + 1}$$
$$\mathcal{L}(1) = \frac{1}{s}$$
$$\mathcal{L}(x) = \frac{1}{s^2}$$
$$\mathcal{L}(f(x - a)\,H(x - a)) = e^{-a\,s}\,\mathcal{L}(f(x)) \quad (140)$$

These give:

$$y(x) = x - \text{Sin}\,x - (x - \text{Sin}(x - 1) - \text{Cos}(x - 1))\,H(x - 1) \quad (141)$$

This is a continuous function with continuous first derivative. The second derivative is of course discontinuous. These facts are reflected in the plots below.

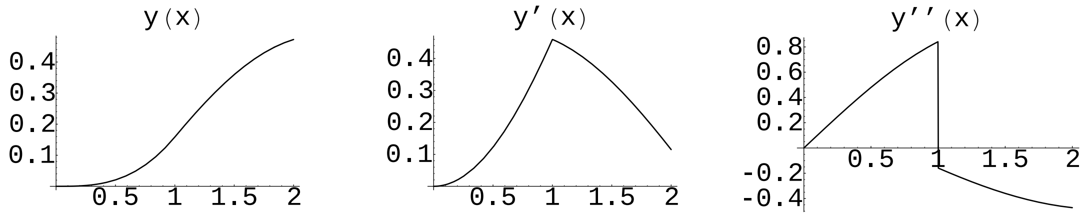


**Figure 7**

# Problem 6 (10 points)

Solve the differential equation

$$\begin{aligned} &y'' + x\,y' - y = 0 \\ &y(0) = 0 \\ &y'(0) = 1 \end{aligned} \tag{142}$$

by taking the Laplace transform of both sides (assuming the transform Y(s) of y(x) exists). Solve the resulting first-order differential equation for Y(s). Be careful to choose the constant of integration so that Y(s) behaves as $s \to \infty$ in a manner consistent with Laplace transforms. Invert Y(s) to find y(x) and check that y(x) satisfies the IVP.

# Solution to Problem 6

$$\begin{aligned} &y'' + x\,y' - y = 0 \\ &y(0) = 0 \\ &y'(0) = 1 \end{aligned} \tag{143}$$

The Laplace transform of $x^n$ f(x) may be computed as follows. First Laplace transform f(x):

$$F(s) = \mathcal{L}\{f(x)\}(s) = \int_0^\infty e^{-sx} f(x)\, dx \tag{144}$$

Then, assuming that f is of exponential order, this integral may be differentiated with respect to s and the derivative passed through the integral:

$$F^{(n)}(s) = \int_0^\infty (-x)^n e^{-sx} f(x)\, dx \tag{145}$$

Finally we have:

$$\boxed{\mathcal{L}\{x^n f(x)\}(s) = (-1)^n \frac{d^n}{ds^n} \mathcal{L}\{f(x)\}(s)} \tag{146}$$

We may now transform the equation:

$$0 = \mathcal{L}\{y'' + xy' - y\} = s^2\, Y - s\, y(0) - y'(0) - \frac{d}{ds}(sY - y(0)) - Y \tag{147}$$

Simplifying gives:

$$\boxed{Y' + \left(\frac{2}{s} - s\right) Y = -\frac{1}{s}} \tag{148}$$

This first order ODE for Y is easily solved using integration factors or the general formula used in previous weeks.

$$Y = \frac{A}{s^2}\, e^{\frac{1}{2} s^2} + \frac{1}{s^2} \tag{149}$$

The Laplace transform of any piecewise continuous function of exponential order can not grow exponentially. Proving this is simple. Suppose f is such a function, then

$$f(x) \le e^{ax} M \tag{150}$$

For some constants M and a. Then we have:

$$|\mathcal{L}(f)(s)| = \left| \int_0^\infty e^{-sx} f(x)\, dx \right| \le \int_0^\infty |e^{-sx}|\, |f(x)|\, dx \tag{151}$$

If s is real, then this bound is:

$$|\mathcal{L}(f)(s)| \le \int_0^\infty e^{-sx} |f(x)|\, dx \le M \int_0^\infty e^{-(s-a)x}\, dx = \frac{M}{s-a} \tag{152}$$

So the Laplace transform of f $\to 0$ as real $s \to \infty$. For this reason, we set A=0 to get:

$$Y = \frac{1}{s^2} \tag{153}$$

Inverting:

$$y(x) = x \tag{154}$$

See appendix for another way to show A=0.

# Problem 7 (5×4 points)

Laplace Transforms $[\mathcal{L}\{f\}](s) = \int_0^\infty \exp(-s\,t)\, f(t)\, dt$ from Taylor series:

a) Show that for n=1,2,3...

$$\mathcal{L}(t^{n-1/2}) = \frac{\sqrt{\pi}\ 1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n\, s^{n+1/2}} \tag{155}$$

b) Find the power series expansion about x=0 for the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2)\, du \tag{156}$$

c) Let $x=\sqrt{z}$ in your series from (b). Take the Laplace transform term by term of the resulting series, to show that for s≥1,

$$\mathcal{L}\left(\operatorname{erf}\left(\sqrt{x}\right)\right) = \frac{1}{s\sqrt{s+1}} \tag{157}$$

d) What happens in part (c) for s<1? Could this equation also be true for s<1? Think about this for a while before checking your answer by continuing with part (e).

e) Show that

$$\frac{d \operatorname{erf}\left(\sqrt{x}\right)}{dx} = \frac{e^{-x}}{\sqrt{\pi x}} \tag{158}$$

Using the integral definition of the Laplace transform given at the beginning of this problem, compute the Laplace transform of the right hand side of this equation. Using the expression for the Laplace transform of a derivative, find the Laplace transform of erf($\sqrt{x}$), and compare to your result in (c). Check your answer to part (d) and discuss

# Solution to Problem 7

a)

From the formula for the derivative of a Laplace transform derived in problem 6 we may write

$$\mathcal{L}(t^{n-1/2}) = \int_0^\infty t^{n-1/2}\, e^{-s\,t}\, dt = (-1)^{n-1} \frac{d^{n-1}}{ds^{n-1}} \int_0^\infty t^{1/2}\, e^{-s\,t}\, dt \tag{159}$$

Let t=$y^2$, integrate once by parts, and use the known expression for the Gaussian integral

$$\int_0^\infty t^{1/2}\, e^{-s\,t}\, dt = \int_0^\infty 2\,y^2\, e^{-s\,y^2}\, dy = \frac{1}{s}\int_0^\infty e^{-s\,y^2}\, dy = \frac{\sqrt{\pi}}{2\,s^{3/2}} \tag{160}$$

This is valid for all Re(s)>0.  So we have:

$$\mathcal{L}(t^{n-1/2}) = (-1)^{n-1} \frac{\sqrt{\pi}}{2} \frac{d^{n-1}}{ds^{n-1}} s^{-3/2} = (-1)^{n-1} \frac{\sqrt{\pi}}{2} \left(\frac{-3}{2}\right)\left(\frac{-5}{2}\right)\ldots\left(\frac{-(2\,n-1)}{2}\right) s^{-n-1/2} = \\ (-1)^{n-1} \frac{\sqrt{\pi}}{2} \frac{(-1)^{n-1}}{2^{n-1}} (3\cdot 5\cdot \ldots\cdot(2\,n-1))\, s^{-n-1/2} = \frac{\sqrt{\pi}\; 1\cdot 3\cdot 5.\,..\,(2\,n-1)}{2^n\; s^{n+1/2}} \tag{161}$$

b)

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du \tag{162}$$

Differentiate this:

$$\mathrm{erf}'(x) = \frac{2}{\sqrt{\pi}}\, e^{-x^2} \tag{163}$$

Using the known series for the exponential, we write:

$$e^{-x^2} = \sum_{n=0}^\infty \frac{(-x^2)^n}{n!} = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\, x^{2\,n} \tag{164}$$

We find a power series for erf(x) by integration (which is permitted since this power series converges uniformly for all x):

$$\mathrm{erf}(x) = \int_0^x \mathrm{erf}'(x)\, dx = \\ \frac{2}{\sqrt{\pi}} \int_0^x \sum_{n=0}^\infty \frac{(-1)^n}{n!}\, x^{2\,n}\, dx = \frac{2}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^x x^{2\,n}\, dx = \frac{2}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n}{n!\,(2\,n+1)}\, x^{2\,n+1} \tag{165}$$

c)

Plug x= $\sqrt{z}$ into our previous sum:

$$\mathrm{erf}\left(\sqrt{z}\right) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n}{n!\,(2\,n+1)} \left(\sqrt{z}\right)^{2\,n+1} = \frac{2}{\sqrt{\pi}} \sum_{n=0}^\infty \frac{(-1)^n}{n!\,(2\,n+1)}\, z^{n+1/2} \tag{166}$$

Now Laplace transform this term by term

$$\mathcal{L}\left(\text{erf}\left(\sqrt{z}\right)\right) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(2n+1)} \mathcal{L}\left(z^{n+1/2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1\cdot 3\cdot 5. ..(2n-1)}{2^n\, s^{n+3/2}} = s^{-3/2} \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5. ..(2n-1)}{n!} \left(\frac{-1}{2s}\right)^n \tag{167}$$

Observe the following:

$$f(y) = (1-y)^{-1/2}$$
$$f^{(n)}(y) = \frac{1\cdot 3\cdot ..\cdot (2n-1)}{2\cdot 2\cdot ..\cdot 2} (1-y)^{-(2n+1)/2} \tag{168}$$

So that we have for |y|<1:

$$(1-y)^{-1/2} = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot ..\cdot (2n-1)}{n!} \left(\frac{y}{2}\right)^n \tag{169}$$

So for |s|>1

$$\mathcal{L}\left(\text{erf}\left(\sqrt{z}\right)\right) = s^{-3/2} \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5. ..(2n-1)}{n!} \left(\frac{-1}{2s}\right)^n = s^{-3/2} \left(1+\frac{1}{s}\right)^{-1/2} = \frac{1}{s\sqrt{1+s}} \tag{170}$$

d)

Recall that we have:

$$\mathcal{L}\left(\text{erf}\left(\sqrt{z}\right)\right) = s^{-3/2} \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5. ..(2n-1)}{n!} \left(\frac{-1}{2s}\right)^n \tag{171}$$

The sum converges provided that

$$\lim_{n\to\infty} \left| \frac{\frac{1\cdot 3\cdot 5... (2n+1)}{(n+1)!} \left(\frac{-1}{2s}\right)^{n+1}}{\frac{1\cdot 3\cdot 5... (2n-1)}{n!} \left(\frac{-1}{2s}\right)^n} \right| < 1 \tag{172}$$

Simplifying this condition gives:

$$\left|\frac{1}{s}\right| < 1 \tag{173}$$

Since we know that power series diverge outside of their radius of convergence, the series we found diverges when |s|<1. As is often the case in complex analysis, formulae containing series which diverge can often be extended to include values outside the radius. For example, the series

$$f(z) = \sum_{n=0}^{\infty} z^n \tag{174}$$

diverges for |z|>1. The function f(z) can be computed for |z|<1

$$f(z) = \frac{1}{1-z} \text{ for } |z| < 1 \tag{175}$$

If we expect that f(z) is analytic (except perhaps at certain poles or branch points), then we might be able to analytically continue f(z) to the whole complex plane (except z=1) to define a new function, the analytic continuation of f(z):

$$F(z) = \frac{1}{1-z} \text{ for } z \neq 1$$ (17 )

We might similarly expect that

$$\mathcal{L}\left(\operatorname{erf}\left(\sqrt{z}\right)\right)(s) \tag{177}$$

is analytic in a larger domain that just |s|>1. So we might be able to define it's analytic continuation to the whole plane (except the pole at s=0, the branch point at s=-1, and some branch cut)

$$F(s) = \frac{1}{s\sqrt{1+s}} \tag{178}$$

In part (e) we will use another method to cover a different part of the plane.

e)

Recall that in part (b) we showed:

$$\frac{d\operatorname{erf}(x)}{dx} = \frac{2}{\sqrt{\pi}} e^{-x^2} \tag{179}$$

By the chain rule we have:

$$\frac{d\operatorname{erf}\left(\sqrt{x}\right)}{dx} = \left(\frac{d\operatorname{erf}\left(\sqrt{x}\right)}{d\sqrt{x}}\right)\left(\frac{d\sqrt{x}}{dx}\right) = \left(\frac{2}{\sqrt{\pi}} e^{-x}\right)\left(\frac{1}{2\sqrt{x}}\right) = \frac{e^{-x}}{\sqrt{\pi x}} \tag{180}$$

We now Laplace transform the right hand side:

$$\mathcal{L}\left(\frac{e^{-x}}{\sqrt{\pi x}}\right) = \int_0^\infty \frac{e^{-x}}{\sqrt{\pi x}} e^{-sx}\, dx = \frac{1}{\sqrt{\pi}} \int_0^\infty x^{-1/2} e^{-(s+1)x}\, dx \tag{181}$$

Make the change of variable $x=y^2$

$$\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-(s+1)y^2}\, dy \tag{182}$$

From the known Gaussian integral, this is defined for Re(s)>-1:

$$\mathcal{L}\left(\frac{e^{-x}}{\sqrt{\pi x}}\right) = \frac{1}{\sqrt{1+s}} \tag{183}$$

We now compute the transform of the derivative:

$$\mathcal{L}\left(\frac{d\operatorname{erf}\left(\sqrt{x}\right)}{dx}\right) = s\,\mathcal{L}\left(\operatorname{erf}\left(\sqrt{x}\right)\right) - \operatorname{erf}(0) = s\,\mathcal{L}\left(\operatorname{erf}\left(\sqrt{x}\right)\right) \tag{184}$$

For s≠0 and Re(s)>-1 this gives:

$$\mathcal{L}\left(\operatorname{erf}\left(\sqrt{x}\right)\right) = \frac{1}{s}\,\mathcal{L}\left(\frac{d\operatorname{erf}\left(\sqrt{x}\right)}{dx}\right) = \frac{1}{s}\,\mathcal{L}\left(\frac{e^{-x}}{\sqrt{\pi x}}\right)(s) = \frac{1}{s\sqrt{1+s}} \tag{185}$$

Which is the same as that found in (c). In part (c) we showed this to be true for |s|>1 and in (d) it's true for Re(s)>-1 and s≠0. These two regions are illustrated below.

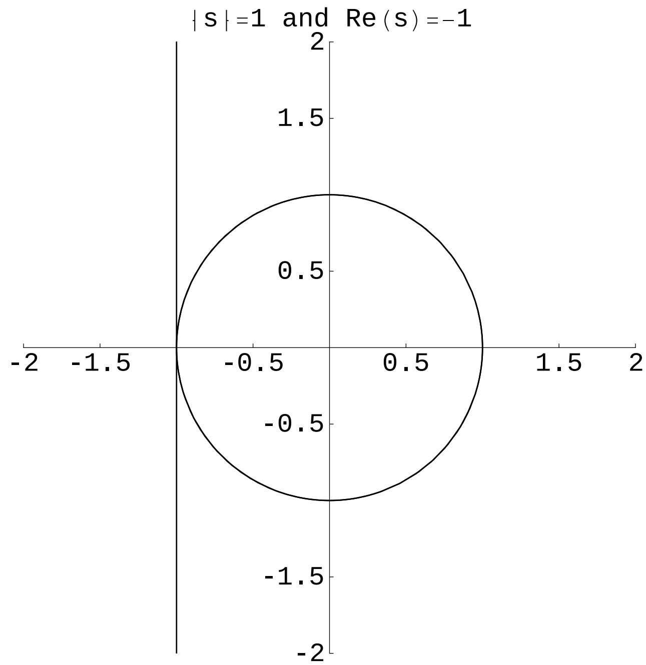


**Figure 8**

You can see how the region A={s: Re(s)>-1 and s≠0} contains some of the region B={s: |s|>1} but that the combination of these two regions (A⋃B) covers the entire complex plane except s=-1 and s=0. So by transforming a function in two different ways we have been able to define the transform for all s≠0,-1.

# Appendix: Watson's Lemma and problem 1

As mentioned above, there is a lemma which makes approximating integrals easier. The basic idea is as follows. For an integral of the form

$$I(x) = \int_a^b f(t)\, e^{-x t}\, dt \tag{186}$$

with b>a≥0 and where:

$$f(t) = (t-a)^{\alpha} \sum_{n=0}^{\infty} a_n (t-a)^{\beta n} \tag{187}$$

is a series for f near t=a, the integral I(x) may be approximated by keeping only the first n terms of the series and integrating over [a,∞):

$$\left| I(x) - \sum_{k=0}^{n} a_k \int_a^{\infty} e^{-x t} (t-a)^{\beta k+\alpha}\, dt \right| << \frac{1}{(x-a)^{1+\beta n+\alpha}} \tag{188}$$

In problem 1 c Watson's Lemma may be applied with a=0, $\alpha$=0, and $\beta$=2 to find:

$$\left| \frac{I(x) - \frac{1}{x}}{\frac{1}{x}} \right| << \frac{1/x}{1/x} = 1 \tag{189}$$

In problem 1 d with a=0, $\alpha$=0, and $\beta$=1 Watson's Lemma may be applied to find:

$$\left| \frac{I(x) - \frac{1}{x}}{\frac{1}{x}} \right| << \frac{1/x}{1/x} = 1 \tag{190}$$

Evidently, in both problems, we should keep more terms to be sure that our approximation is accurate enough. For example, in problem 1d we keep two terms and Watson's lemma tell us that

$$\left|\frac{I(x) - \frac{1}{x} + \frac{1}{x^2}}{\frac{1}{x} - \frac{1}{x^2}}\right| << \frac{1/x^3}{\frac{1}{x} - \frac{1}{x^2}} = \frac{1}{x^2 - x} << \frac{1}{100^2 - 100} = 0.0001010 \qquad (191)$$

Without using Watson's Lemma, it is still sometimes easy to find your own bounds. For example, consider 1c. Let us try to bound the error.

$$\left|I(x) - \frac{1}{x}\right| = \left|\int_0^\infty \left(\frac{e^{-xt}}{1+t^2} - e^{-xt}\right) dt\right| = \left|\int_0^\infty \frac{-t^2 e^{-xt}}{1+t^2} dt\right| = \int_0^\infty \frac{t^2 e^{-xt}}{1+t^2} dt \le \int_0^\infty e^{-xt} dt = \frac{1}{x} \qquad (192)$$

So we have:

$$\left|\frac{I(x) - \frac{1}{x}}{\frac{1}{x}}\right| \le 1 \qquad (193)$$

Compare this to using Watson's lemma where we found the slightly better bound

$$\left|\frac{I(x) - \frac{1}{x}}{\frac{1}{x}}\right| << 1 \qquad (194)$$

# Appendix: Showing that A=0 in problem 6

$$Y = \frac{A}{s^2} e^{\frac{1}{2} s^2} + \frac{1}{s^2} \qquad (195)$$

Another way of showing that A=0 gives the correct answer is to write:

$$y(x) = \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{A}{s^2} e^{\frac{1}{2} s^2} + \frac{1}{s^2}\right) = A\, g(x) + x \qquad (196)$$

For some non-trivial g(x). Now, you may check that g(x) must satisfy:

$$\begin{aligned} &g'' + x g' - g = 0 \\ &g(0) = 0 \\ &g'(0) = 0 \end{aligned} \qquad (197)$$

This linear equation has analytic coefficients and by the Cauchy-Kowalevski theorem (F. John, pg74) must have a unique solution which is analytic near x=0. Since every analytic function has a unique power series representation we have:

$$g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n \qquad (198)$$

But we find:

$$\begin{aligned} &g(0) = 0 \\ &g'(0) = 0 \\ &g''(0) = (g - x g')_{x=0} = 0 \\ &g'''(0) = ((g - x g')')_{x=0} = 0 \\ &\text{etc.} \end{aligned} \qquad (199)$$

So that any derivative of g at x=0 vanishes. This means that the power series for g has all zero cofficients. Hence g(x)=0 in some neighborhood of 0. By analytic continuation we must have g(x)=0 for all x. But this is the same result as you get by setting A=0.