Problem Set 3

January 23, 2004 — Due January 30, 2004
ACM 95b/100b — 3pm in Firestone 303
E. Sterl Phinney — (2 pts) Include grading section number

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1. (4×5 points) The following (trivial once you 'get it') problem is designed to help those of you who had trouble with Problem Set 1's problem 7c. It will also help you find asymptotic expansions of Laplace integrals and their relatives.

a) Evaluate

$$\int_0^a \exp(-1000t)\, dt \tag{1}$$

for $a = 10^{-4}$, 10^{-3}, 3×10^{-3}, 9×10^{-3}, 0.1, 10 and ∞. Also give your answers in fixed decimal form to 7 digits [i.e. numbers like 0.1234567], and explain any trend you notice.

b) Now consider the more general integral

$$I(a) = \int_0^a \exp(-xt)\, dt \tag{2}$$

for $x \gg 100$. What range of values of a makes $I(a) = I(\infty)(1+\varepsilon)$, with $|\varepsilon| < 0.01$ (i.e. gives $I(\infty)$ to 1% accuracy)? Your answer may depend on x.

c) Use reasoning motivated by part (b) to find the simplest function of x which approximates

$$I(x) = \int_0^\infty \frac{e^{-xt}}{(1+t^2)}\, dt \tag{3}$$

to 1% accuracy for $x \gg 100$. Justify your error estimate.

d) Use reasoning motivated by part (b) to find the simplest function of x which approximates

$$I(x) = \int_0^{1/5} \frac{e^{-xt}}{\sqrt{2t + t^3/4 + \cos t}}\, dt \tag{4}$$

to 1% accuracy for $x \gg 100$. Justify your error estimate.

2. (5 points) Prove that for an analytic function $f(t)$ with Laplace transform $F(s)$,

$$\lim_{s\to\infty} sF(s) = \lim_{t\to 0+} f(t)\ . \tag{5}$$

3. (6×4 points) Fun with Dirac

a) Show that

$$x \frac{d}{dx} \delta(x) = -\delta(x) \; . \tag{6}$$

(Hint: use the Gaussian δ sequence, $\delta_n = n/\sqrt{\pi} \exp(-n^2 x^2)$; integration by parts may also be helpful.)

b) Show that for $f'(x)$ continuous at $x = 0$,

$$\int_{-\infty}^{\infty} \delta'(x) f(x) dx = -f'(0) \tag{7}$$

c) Show that if x_0 is the solution of $f(x_0) = 0$,

$$\delta(f(x)) = \left| \frac{df(x)}{dx} \right|^{-1} \delta(x - x_0) \tag{8}$$

d) What happens in (c) if $f(x)$ has more than one zero? Use your result to find for $x_1 \neq x_2$

$$\delta[(x - x_1)(x - x_2)] \; . \tag{9}$$

e) Consider the three dimensional delta function in cartesian coordinates:

$$\delta(\vec{\mathbf{r}} - \vec{\mathbf{r_0}}) \equiv \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \; . \tag{10}$$

Introduce a new coordinate system with three new coordinates ξ, η and ζ by $x = X(\xi, \eta, \zeta)$, $y = Y(\xi, \eta, \zeta)$, $z = Z(\xi, \eta, \zeta)$. Show that the equation of part (c) generalizes to

$$\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) = \delta(\xi - \xi_0)\delta(\eta - \eta_0)\delta(\zeta - \zeta_0) \left| \frac{\partial(X, Y, Z)}{\partial(\xi, \eta, \zeta)} \right|^{-1} \tag{11}$$

where $\partial(X, Y, Z)/\partial(\xi, \eta, \zeta)$ is the Jacobian determinant of partial derivatives of X, Y, Z with respect to ξ, η, ζ, and ξ_0, η_0 and ζ_0 are the solutions of $x_0 = X(\xi_0, \eta_0, \zeta_0)$, $y_0 = Y(\xi_0, \eta_0, \zeta_0)$, $z_0 = Z(\xi_0, \eta_0, \zeta_0)$.

f) In particular, show that in cylindrical coordinates R, ϕ, z

$$\delta(\vec{\mathbf{R}} - \vec{\mathbf{R_0}}) = (1/R)\delta(R - R_0)\delta(\phi - \phi_0)\delta(z - z_0) \; , \tag{12}$$

in spherical polar coordinates r, θ, ϕ (θ being the polar angle),

$$\delta(\vec{\mathbf{r}} - \vec{\mathbf{r_0}}) = 1/(r^2 \sin\theta)\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0) \; , \tag{13}$$

and if $\mu = \cos(\theta)$ is substituted in spherical polar coordinates,

$$\delta(\vec{\mathbf{r}} - \vec{\mathbf{r_0}}) = 1/(r^2)\delta(r - r_0)\delta(\mu - \mu_0)\delta(\phi - \phi_0) \; . \tag{14}$$

You may use a computer algebra package, or look up the Jacobians if you view manual algebra or cleverness as too challenging.

4. (10 points) Laplace operations on discontinuous functions

a) (4 points) Consider the function

$$\phi(t) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \\ \phi_0, & t = t_0 \end{cases} \tag{15}$$

Compute the Laplace transform $\Phi(s)$ of $\phi(t)$. Then by explicit integration of the Mellin inversion formula along the Bromwich contour (being careful about principal values when needed), show that the inverse Laplace transform of $\Phi(s)$ is the Heaviside step function

$$\{\mathcal{L}^{-1}\Phi\}(t) = H(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \\ 1/2, & t = t_0 \end{cases} \tag{16}$$

and thus need not agree with with $\phi(t)$ at t_0.

b) (2 points) Is this consistent with Lerch's theorem (as stated in class: if $f_1(t)$ and $f_2(t)$ have the same Laplace transform, then f_1 and f_2 differ by a null function, i.e. $f_1 - f_2 = N(t)$, where $\int_0^{t_0} N(t)\,dt = 0$ for all $t_0 > 0$.)?

c) (4 points) Now consider

$$\psi(t) = \begin{cases} f(t), & t < t_0 \\ g(t), & t > t_0 \ . \end{cases} \tag{17}$$

By considering the continuous (except possibly at the isolated point t_0) function

$$\psi(t) - H(t - t_0)\left(g(t_0^+) - f(t_0^-)\right) , \tag{18}$$

find $[\mathcal{L}^{-1}\{\mathcal{L}(\psi)\}](t_0)$.

5. (2 × 7 points) Use the shifting theorems to assist you in solving the following initial value problems (you may in addition use Laplace transform tables —e.g. the one handed out in class):

a)

$$4y'' - 4y' + 37y = 0, \quad y(0) = 3, \quad y'(0) = 3/2. \tag{19}$$

b)

$$y'' + y = r(t), \quad y(0) = 0, \quad y'(0) = 0, \quad r(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & t > 1 \end{cases} \tag{20}$$

6. (10 points) Solve the differential equation

$$y'' + xy' - y = 0, \quad y(0) = 0, \quad y'(0) = 1 \tag{21}$$

by taking the Laplace transform of both sides (assuming the transform $\hat{y}(s)$ of $y(x)$ exists). Solve the resulting first-order differential equation for $\hat{y}(s)$. Be careful to choose the constant of integration so that $\hat{y}(s)$ behaves as $s \to \infty$ in a manner consistent with Laplace transforms. Invert $\hat{y}(s)$ to find $y(x)$, and check that $y(x)$ satisfies the IVP.

7. (5×4 points) Laplace Transforms $[\mathcal{L}\{f\}](s) = \int_0^\infty \exp(-st)f(t)dt$ from Taylor series:
a) Show that for $n = 1, 2, 3 \ldots$,

$$\mathcal{L}(t^{(n-1/2)}) = \frac{\sqrt{\pi}}{2^n} \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{s^{(n+1/2)}} \tag{22}$$

b) Find the power series expansion about $x = 0$ for the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2)\, du \tag{23}$$

c) Let $x = \sqrt{z}$ in your series from (b). Take the Laplace Transform term by term of the resulting series, to show that for $s \geq 1$,

$$\mathcal{L}\{\mathrm{erf}(\sqrt{x})\} = \frac{1}{s\sqrt{s+1}} \tag{24}$$

d) What happens in part (c) for $s < 1$? Could equation (24) also be true for $s < 1$? Think about this for a while before checking your answer by continuing with part (e).

e) Show that

$$\frac{d\,\mathrm{erf}(\sqrt{x})}{dx} = \frac{e^{-x}}{\sqrt{\pi x}} \tag{25}$$

Using the integral definition of the Laplace transform given at the beginning of this problem, compute the Laplace transform of the right hand side of equation (25). Using the expression for the Laplace transform of a derivative, find the Laplace transform of $\mathrm{erf}(\sqrt{x})$, and compare to your result in (c). Check your answer to part (d), and discuss.

Total points: 105