ACM95/100b
Problem Set 2 Solutions
01/18/04

Problem 1 (3×7 points)

Solve the following linear initial value problems. Sketch the solution and describe the behavior as the independent variable $x \to \infty$.

a)

$$y'' + 6y' + 5y = 0 \quad y(0) = 3 \quad y'(0) = -2 \tag{1}$$

b)

$$y'' + 2y' + 5y = 0 \quad y(0) = 0 \quad y'(0) = 5 \tag{2}$$

c)

$$y'' - 2y' + y = 0 \quad y(0) = 1 \quad y'(0) = 0 \tag{3}$$

Solution to Problem 1

In general, an n^{th} order homogeneous linear constant coefficient ODE has n linearly independent solutions. The general solution is then of the form

$$y = \sum P_i(x) e^{r_i x} \tag{4}$$

Where P are polynomials and r are constants. The easiest way to begin constructing these solutions is to start with the ODE:

$$y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_1 y' + a_0 y = 0 \tag{5}$$

And make the following guess:

$$y = e^{rx} \tag{6}$$

Plugging this in and simplifying gives an nth degree polynomial equation for r:

$$r^n + a_{n-1} r^{n-1} + ... + a_1 r + a_0 = 0 \tag{7}$$

After factoring this becomes:

$$(r - r_1)^{m_1} ... (r - r_k)^{m_k} = 0 \tag{8}$$

To each solution of this equation corresponds a solution to the ODE of the form:

$$y_i = P_i(x) e^{r_i x} \tag{9}$$

where P_i is an $(m_i - 1)^{th}$ degree polynomial.

a)

$$y'' + 6y' + 5y = 0 \quad y(0) = 3 \quad y'(0) = -2 \tag{10}$$

We guess y= $e^{r x}$. This gives:

$$0 = r^2 + 6 r + 5 = (r + 1) (r + 5) \tag{11}$$

So our solution is of the form:

$$y = A e^{-x} + B e^{-5 x} \tag{12}$$

Fitting the initial data gives:

$$\begin{aligned} 3 &= y(0) = A + B \\ 2 &= -y'(0) = A + 5 B \end{aligned} \tag{13}$$

Solving this system (either by Gaussian elimination or matrix inversion) gives

$$\begin{aligned} A &= 13/4 \\ B &= -1/4 \end{aligned} \tag{14}$$

So the solution to the initial value problem is:

$$y = \frac{13 e^{-x} - e^{-5 x}}{4} \tag{15}$$

Since each of the exponents is negative, as x→∞, y→0 monotonically (i.e. the slope is always negative). This is seen in the following plot

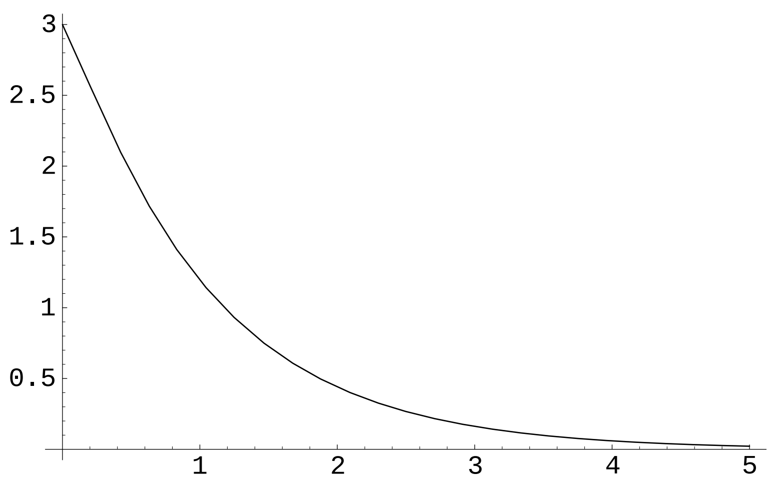

Figure 1

b)

$$\begin{aligned} &y'' + 2 y' + 5 y = 0 \\ &y(0) = 0 \\ &y'(0) = 5 \end{aligned} \tag{16}$$

We guess y= $e^{r x}$. This gives:

$$0 = r^2 + 2 r + 5 = (r - (-1 + 2 i)) (r - (-1 - 2 i)) \tag{17}$$

So our solution is of the form:

$$y = A e^{(-1+2 i) x} + B e^{(-1-2 i) x} \tag{18}$$

Although it would be fine to proceed to find the coefficients with the solution in this form, since our solution will be a real function, we use the following result of Euler's:

$$e^{\theta i} = \text{Cos}\,\theta + i\,\text{Sin}\,\theta \tag{19}$$

This allows us to write:

$$A\, e^{(-1+2\,i)\,x} + B\, e^{(-1-2\,i)\,x} = e^{-x}\,(a \,\mathrm{Cos}\, 2\,x + b\, \mathrm{Sin}\, 2\,x) \qquad (20)$$

for some constants a and b.

Fitting the initial data gives:

$$\begin{aligned} 0 &= y\,(0) = a \\ 5 &= y'\,(0) = 2\,b - a \end{aligned} \qquad (21)$$

Solving this system gives

$$\begin{aligned} a &= 0 \\ b &= 5/2 \end{aligned} \qquad (22)$$

So the solution to the initial value problem is:

$$y = \frac{5}{2}\, e^{-x}\, \mathrm{Sin}\, 2\,x \qquad (23)$$

Since the exponent is negative, as $x \to \infty$, $y \to 0$. The Sine term causes the solution to oscillate while the exponential term steadily decreases the amplitude of this oscillation. This is seen in the following plot

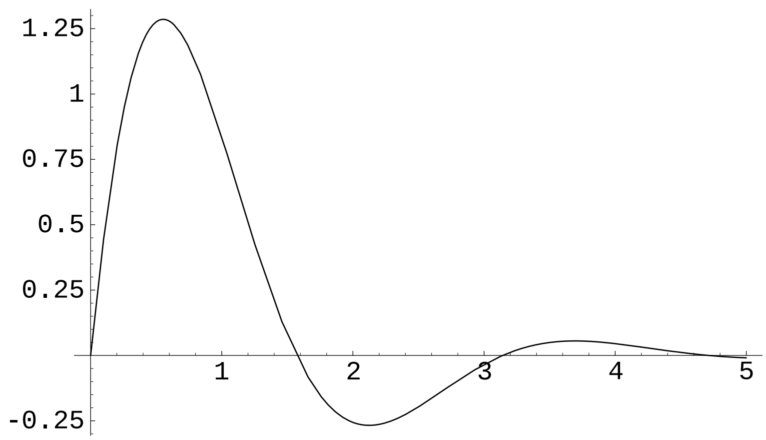

Figure 2

c)

$$\begin{aligned} & y'' - 2\,y' + y = 0 \\ & y\,(0) = 1 \\ & y'\,(0) = 0 \end{aligned} \qquad (24)$$

We guess $y = e^{r\,x}$. This gives:

$$0 = r^2 - 2\,r + 1 = (r-1)^2 \qquad (25)$$

From the general results derived above, since the multiplicity of the root is 2, our solution is of the form:

$$y = (A + B\,x)\, e^{x} \qquad (26)$$

Fitting the initial data gives:

$$\begin{aligned} 1 &= y\,(0) = A \\ 0 &= y'\,(0) = A + B \end{aligned} \qquad (27)$$

Solving this system gives

$$A = 1$$
$$B = -1 \quad (28$$

So the solution to the initial value problem is:

$$y = (1 - x)\, e^{x} \quad (29)$$

Since the exponent is positive, and 1-x<0 for x>1, as x→∞ we have y→-∞. This is seen clearly in the following plot:

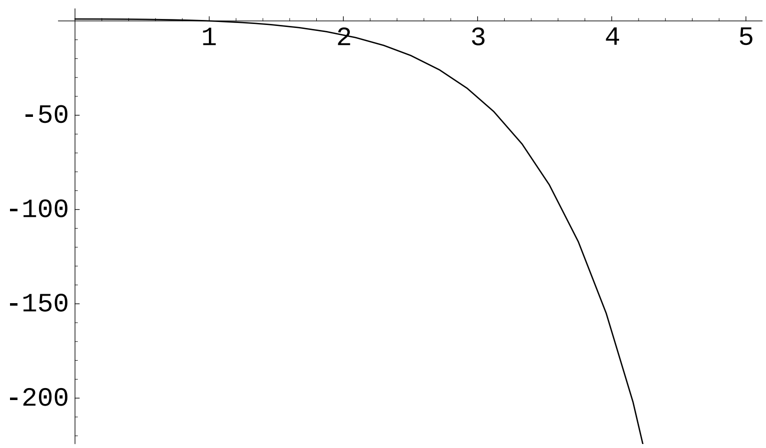

Figure 3

Problem 2 (2×7 points)

Consider the Legendre equation of order one (the Legendre equation arises when separating Laplace's equation in spherical coordinates -e.g. in electromagnetics, quantum mechanics and potential theory of spherical bodies -e.g. planets)

$$(1 - x^2)\, y''(x) - 2\, x\, y'(x) + 2\, y(x) = 0$$
$$x \in (-1, 1) \quad (30)$$

a) find (up to a constant multiplicative factor) the Wronskian of two solutions without solving the ODE

b) verify that y(x)=x is a solution and use reduction of order to find a general solution to the ODE

Solution to Problem 2

Suppose we are given a 2nd order linear homogeneous ODE as well as two solutions of it:

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = 0$$
$$p(x)\, z'' + q(x)\, z' + r(x)\, z = 0 \quad (31)$$

If we multiply these by z and y respectively and then subtract them we get:

$$0 = z\,(p\, y'' + q\, y' + r\, y) - y\,(p\, z'' + q\, z' + r\, z) =$$
$$p\,(z\, y'' - y\, z'') + q\,(z\, y' - y\, z') = p\,(z\, y' - y\, z')' + q\,(z\, y' - y\, z') \quad (32)$$

The quantity appearing in the last two terms in parenthesis is known as the Wronskian. Evidently, the Wonskian, W(x)=z(x)y'(x)-y(x)z'(x), is a solution to the following ODE:

$$p(x)\, W' + q(x)\, W = 0 \quad (33)$$

This is easily solved by using integrating factors or the general solution formula for first order linear ODE (see last week's homework). Either way we find:

$$W = C\, e^{-\int \frac{q(x)}{p(x)}\, dx} \quad (34)$$

For some constant C that can be fixed only if we know some initial or boundary conditions for y and z.

a)

Suppose we have two solutions to the ODE:

$$\begin{aligned}(1-x^2)\,y'' - 2\,x\,y' + 2\,y = 0\\(1-x^2)\,z'' - 2\,x\,z' + 2\,z = 0\end{aligned} \tag{35}$$

Following the procedure shown above, we multiply by z and y respectively, subtract, and simplify to get:

$$(1-x^2)\,(z\,y' - y\,z') - 2\,x\,(\,z\,y' - y\,z') = 0 \tag{36}$$

The term in parenthesis is the Wronskian. We write:

$$0 = (1-x^2)\,W - 2\,x\,W = ((1-x^2)\,W)' \tag{37}$$

Integrating gives:

$$(1-x^2)\,W = C \tag{38}$$

If x≠±1, then we have an expression for the Wronskian up to an arbitrary constant determined by the specific form of y and z.

$$W = \frac{C}{1-x^2} \tag{39}$$

b)

Plug y(x)=x into the equation:

$$(1-x^2)\,(x)'' - 2\,x\,(x)' + 2\,x = (1-x^2)\,(0) - 2\,x\,(1) + 2\,x = 0 \tag{40}$$

Knowing that y(x)=x is a solution, we can now take two approaches.

(i) The reduction of order technique exploits the fact that if y=f(x) is any solution to an nth order linear homogeneous ODE, then y=f(x)u(x) can be plugged into the ODE resulting in a new ODE which is of order n-1. (This is a special case of Lie's general order reduction techniques using symmetry groups of differential equations)

To be explicit, since y=x solves the Legendre equation, we look for a solution of the form y=xu(x). Plugging this in gives:

$$\begin{aligned}0 = (1-x^2)\,(x\,u)'' - 2\,x\,(x\,u)' + 2\,x\,u =\\(1-x^2)\,(2\,u' + x\,u'') - 2\,x\,(u + x\,u') + 2\,x\,u = (1-x^2)\,x\,u'' + (2-4\,x^2)\,u'\end{aligned} \tag{41}$$

This may appear to be a second order ODE,but setting v=u' gives:

$$(1-x^2)\,x\,v' + (2-4\,x^2)\,v = 0 \tag{42}$$

We write this as

$$v' + \frac{2-4\,x^2}{(1-x^2)\,x}\,v = 0 \tag{43}$$

An integrating factor is

$$I = e^{\int \frac{2-4\,x^2}{(1-x^2)\,x}\,dx} = x^2\,(x^2-1) \tag{44}$$

This allows us to write:

$$(x^2\,(x^2-1)\,v)' = 0 \tag{45}$$

Which gives:

$$v = \frac{A}{x^2\,(x^2-1)} \tag{46}$$

We then integrate to find u(x)

$$u = \int v\, dx = \int \frac{A}{x^2\,(x^2-1)}\, dx = B + A\left(\frac{1}{x} + \frac{1}{2}\ln\left|\frac{1-x}{1+x}\right|\right) \tag{47}$$

So the general solution is of the form:

$$y = x\,u = B\,x + A\left(1 + \frac{1}{2}\,x\ln\left|\frac{1-x}{1+x}\right|\right) \tag{48}$$

(ii) Instead of using the reduction of order technique, we could find a lower order ODE for the general solution just by examining the Wronskian:

$$y\,z' - z\,y' = \frac{C}{1-x^2} \tag{49}$$

Set y=x , then for x≠0 we have:

$$z' - \frac{1}{x}\,z = \frac{C}{x\,(1-x^2)} \tag{50}$$

An integrating factor is

$$I = e^{\int \frac{-1}{x}\,dx} = \frac{1}{x} \tag{51}$$

This gives:

$$\left(\frac{1}{x}\,z\right)' = \frac{C}{x^2\,(1-x^2)} \tag{52}$$

Integrating once and simplifying gives us:

$$z = B - C\left(\frac{1}{x} + \frac{1}{2}\ln\left|\frac{1-x}{1+x}\right|\right) \tag{53}$$

With A=-C this is the same as we found using the reduction of order method.

Problem 3 (10 points)

[Variation of parameters.] Find a particular solution to

$$\begin{aligned} &t^2\,y'' - 2\,y = 3\,t^2 - 1 \\ &t > 0 \end{aligned} \tag{54}$$

Hint: notice that $y_1(t)=t^2$ and $y_2(t)=t^{-1}$ are linearly independent solutions to the homogeneous problem.

Solution to Problem 3

Suppose we have two linearly independent solutions to a linear homogeneous second order ODE

$$\begin{aligned} &u'' + p\,u' + q\,u = 0 \\ &v'' + p\,v' + q\,v = 0 \end{aligned} \tag{55}$$

If we wish to find a particular solution to the corresponding inhomogeneous equation

$$y'' + p\,y' + q\,y = r \tag{56}$$

the variation of parameters technique says we should seek a solution of the form y=au+bv where a and b are as yet undetermined functions of x. (Lie's methods derived from the general theory of symmetry groups give the variation of parameters technique as a special case of a broader method)
Making the guess y=au+bv gives:

$$(a\,u + b\,v)'' + p\,(a\,u + b\,v)' + q\,(a\,u + b\,v) = r \tag{57}$$

Through some manipulation, this can be put in the form:

$$a\,(u'' + p\,u' + q\,u) + b\,(v'' + p\,v' + q\,v) + (a'\,u + b'\,v)' + p\,(a'\,u + b'\,v) + (a'\,u' + b'\,v') = r \tag{58}$$

The first two terms in parenthesis vanish since u and v are solutions of the homogeneous problem. This leaves:

$$(a'\,u + b'\,v)' + p\,(a'\,u + b'\,v) + (a'\,u' + b'\,v') = r \tag{59}$$

Since we have two functions, a and b, to prescribe, we use one of these degrees of freedom to require $a'u+b'v=0$. This makes the first two terms in parenthesis vanish, leaving only:

$$a'\,u' + b'\,v' = r \tag{60}$$

We then have a system of equations for a' and b':

$$\begin{aligned} a'\,u + b'\,v &= 0 \\ a'\,u' + b'\,v' &= r \end{aligned} \tag{61}$$

Or written in matrix form this is:

$$\begin{pmatrix} u & v \\ u' & v' \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix} \tag{62}$$

The matrix will be invertible if the Wronskian of u and v is non-zero.

For our problem we have:

$$\begin{aligned} &y'' - \frac{2}{t^2}\,y = 3 - \frac{1}{t^2} \\ &u = t^2 \\ &v = 1/t \end{aligned} \tag{63}$$

The Wronskian of u and v is:

$$W = u\,v' - v\,u' = -3 \neq 0 \tag{64}$$

So we have a particular solution of the form:

$$y = a\,t^2 + b\,\frac{1}{t} \tag{65}$$

With

$$\begin{pmatrix} t^2 & 1/t \\ 2\,t & -1/t^2 \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 0 \\ 3 - 1/t^2 \end{pmatrix} \tag{66}$$

Inverting the matrix gives:

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \frac{1}{-3} \begin{pmatrix} -1/t^2 & -1/t \\ -2\,t & t^2 \end{pmatrix} \begin{pmatrix} 0 \\ 3 - 1/t^2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3/t - 1/t^3 \\ 1 - 3\,t^2 \end{pmatrix} \tag{67}$$

We integrate for $t>0$ to find:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 3\ln t + 1/2\,t^2 \\ t - t^3 \end{pmatrix} \tag{68}$$

Where α and β are arbitrary constants.

So our general solution for $t>0$ is:

$$y = a\,u + b\,v = \alpha\, t^2 + \beta\, \frac{1}{t} + t^2 \left(\ln t - \frac{1}{3}\right) + \frac{1}{2} \tag{69}$$

A particular solution is found by choosing any values for α and β.

Problem 4 (10 points)

[Variation of parameters] Show that the solution of the IVP

$$\begin{aligned} &y'' + y = g(x) \\ &y(0) = 0 \\ &y'(0) = 0 \end{aligned} \tag{70}$$

has the form

$$y = \int_0^x \sin(x - s)\, g(s)\, ds \tag{71}$$

Solution to Problem 4

$$\begin{aligned} &y'' + y = g(x) \\ &y(0) = 0 \\ &y'(0) = 0 \end{aligned} \tag{72}$$

Using the results derived in the solution to problem 3, the general solution to this problem will be of the form:

$$y = a\,u + b\,v \tag{73}$$

Where u and v are any two linearly independent solutions of

$$y'' + y = 0 \tag{74}$$

and a and b are functions given by the system of first order equations

$$\begin{pmatrix} u & v \\ u' & v' \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix} \tag{75}$$

To find solutions to the homogeneous problem, proceed as in problem 1 by guessing solutions of the form $e^{r x}$. This gives

$$0 = r^2 + 1 = (r + i)(r - i) \tag{76}$$

So there are solutions of the form:

$$y = A\, e^{i x} + B\, e^{-i x} = C \operatorname{Sin} x + D \operatorname{Cos} x \tag{77}$$

Let's take u=Sin x and v=Cos x. A quick check shows that the Wronskian is:

$$W = (\operatorname{Sin} x)(\operatorname{Cos} x)' - (\operatorname{Sin} x)'(\operatorname{Cos} x) = -(\operatorname{Sin}^2 x + \operatorname{Cos}^2 x) = -1 \tag{78}$$

This means that we may invert the system:

$$\begin{pmatrix} \operatorname{Sin} x & \operatorname{Cos} x \\ \operatorname{Cos} x & -\operatorname{Sin} x \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix} \tag{79}$$

We get:

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} \operatorname{Sin} x & \operatorname{Cos} x \\ \operatorname{Cos} x & -\operatorname{Sin} x \end{pmatrix} \begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} g \operatorname{Cos} x \\ -g \operatorname{Sin} x \end{pmatrix} \tag{80}$$

The general solution is then:

$$y = a\,u + b\,v = \left(A + \int g(x) \operatorname{Cos} x\, dx\right) \operatorname{Sin} x + \left(B + \int -g(x) \operatorname{Sin} x\, dx\right) \operatorname{Cos} x \tag{81}$$

To apply the initial data, it is more convenient to use definite integrals instead of indefinite ones. The resulting expression is still the general solution to the ODE.

$$y = \left(A + \int_0^x g(t) \operatorname{Cos} t \, dt\right) \operatorname{Sin} x + \left(B - \int_0^x g(t) \operatorname{Sin} t \, dt\right) \operatorname{Cos} x \tag{82}$$

The initial data now gives:

$$\begin{aligned} 0 &= y(0) = B \\ 0 &= y'(0) = A \end{aligned} \tag{83}$$

So the solution to our initial value problem is:

$$y = \operatorname{Sin} x \int_0^x g(t) \operatorname{Cos} t \, dt - \operatorname{Cos} x \int_0^x g(t) \operatorname{Sin} t \, dt \tag{84}$$

Since the integration variable is t, the Sin x and Cos x terms can be brought under the integral sign and both integrals can be combined into one.

$$y = \int_0^x g(t) (\operatorname{Sin} x \operatorname{Cos} t - \operatorname{Cos} x \operatorname{Sin} t) \, dt \tag{85}$$

We now apply a trigonometric identity:

$$\operatorname{Sin} x \operatorname{Cos} t - \operatorname{Cos} x \operatorname{Sin} t = \operatorname{Sin}(x - t) \tag{86}$$

So our solution is most compactly written as:

$$y = \int_0^x \operatorname{Sin}(x - t) \, g(t) \, dt \tag{87}$$

Problem 5 (4×7 points)

Use the method of undetermined coefficients (See note below) to find the general solution of the ODE describing a driven harmonic oscillator

$$L(y) = y'' + y = x^2 + e^{-ax} \operatorname{Sin}(\omega x) \tag{88}$$

a) for real ω and real $a \neq 0$

b) for $a=0$ and $\omega=1$. What is special about this case?

c) Set $\omega=1$ and let $a \to 0$ in your answer to (a). Show that this reproduces your answer to (b).

d) Can you also reproduce your answer to (b) by taking the limit in the other order (i.e. setting $a=0$ and letting $\omega \to 1$)?

Note for parts (a,b): you may wish to try the method of undetermined coefficients (notice that the linear differential operator L applied to polynomials, polynomials times exponentials and polynomials times trig functions, returns functions in those same classes. So for a sufficiently general choice of 'guessed solution' consisting of sums of such terms, the coefficients in those sums can be chosen so the sum satisfies the ODE).

Solution to Problem 5

$$y'' + y = x^2 + e^{-ax} \operatorname{Sin}(\omega x) \tag{89}$$

The homogeneous problem has solutions of the form Sin(x) and Cos(x). The inhomogeneous part has polynomial terms as well as trig functions combined with exponentials. For $a \neq 0$,the most general solution to this equation will then be of the form:

$$y = A \operatorname{Sin} x + B \operatorname{Cos} x + C + D x + E x^2 + F e^{-ax} \operatorname{Sin}(\omega x) + G e^{-ax} \operatorname{Cos}(\omega x) \tag{90}$$

Since the A and B terms correspond to homogeneous solutions, we simply take the remaining part and plug into the ODE:

$$(C + D x + E x^2 + F e^{-ax} \text{Sin}(\omega x) + G e^{-ax} \text{Cos}(\omega x))'' + (C + D x + E x^2 + F e^{-ax} \text{Sin}(\omega x) + G e^{-ax} \text{Cos}(\omega x)) = x^2 + e^{-a x} \text{Sin}(\omega x) \quad (91$$

Computing derivatives and rearranging gives:

$$(C + 2 E) + (D) x + (E - 1) x^2 + ((1 + a^2 - \omega^2) G - 2 a F \omega) e^{-ax} \text{Cos}\, x + ((1 + a^2 - \omega^2) F + 2 a G \omega - 1) e^{-ax} \text{Sin}\, x = 0 \quad (92)$$

In order for this expression to be zero, each of the terms in parenthesis must be zero. We find:

$$\begin{aligned} C &= -2 \\ D &= 0 \\ E &= 1 \\ F &= \frac{1 + a^2 - \omega^2}{4 a^2 \omega^2 + (1 + a^2 - \omega^2)^2} \\ G &= \frac{2 a\omega}{4 a^2 \omega^2 + (1 + a^2 - \omega^2)^2} \end{aligned} \quad (93)$$

So the general solution for a≠0 is:

$$y = A \text{Sin}\, x + B \text{Cos}\, x - 2 + x^2 + \frac{1 + a^2 - \omega^2}{4 a^2 \omega^2 + (1 + a^2 - \omega^2)^2} e^{-ax} \text{Sin}(\omega x) + \frac{2 a\omega}{4 a^2 \omega^2 + (1 + a^2 - \omega^2)^2} e^{-ax} \text{Cos}(\omega x) \quad (94)$$

b)

When a=0 and ω=1, the exponential/trig term in the inhomogeneity is reduced to Sin(x). So the inhomogeneity contains a Sin(x) term which is not linearly independent of the solutions to the homogeneous ODE (i.e. the oscillator is being forced at resonance). This motivates us to seek a solution of the form:

$$y = (A + C x) \text{Sin}\, x + (B + D x) \text{Cos}\, x + E + F x + G x^2 \quad (95)$$

Intuitively, if a linear oscillator is forced at resonance, we should expect it to oscillate with increasing amplitude. The A and B terms correspond to the homogeneous solution, so we take what remains and insert it into the ODE:

$$(C x \text{Sin}\, x + D x \text{Cos}\, x + E + F x + G x^2)'' + (C x \text{Sin}\, x + D x \text{Cos}\, x + E + F x + G x^2) = x^2 + \text{Sin}\, x \quad (96)$$

Differentiating and simplifying gives:

$$(2 G + E) + (F) x + (G - 1) x^2 + (2 C) \text{Cos}\, x - (1 + 2 D) \text{Sin}\, x = 0 \quad (97)$$

For this to hold for all x, we need each term in parenthesis to be zero. This gives us the coefficients:

$$\begin{aligned} C &= 0 \\ D &= -1/2 \\ E &= -2 \\ F &= 0 \\ G &= 1 \end{aligned} \quad (98)$$

So the general solution to the ODE is:

$$y = A \text{Sin}\, x + B \text{Cos}\, x - 2 + x^2 - \frac{1}{2} x \text{Cos}\, x \quad (99)$$

c)

Recall the solution from part (a):

$$y = A\,\mathrm{Sin}\,x + B\,\mathrm{Cos}\,x - 2 + x^2 + \frac{1 + a^2 - \omega^2}{4 a^2 \omega^2 + (1 + a^2 - \omega^2)^2} e^{-ax} \mathrm{Sin}(\omega x) + \frac{2 a\omega}{4 a^2 \omega^2 + (1 + a^2 - \omega^2)^2} e^{-ax} \mathrm{Cos}(\omega x) \quad (100)$$

Setting ω=1 gives:

$$y = A\,\mathrm{Sin}\,x + B\,\mathrm{Cos}\,x - 2 + x^2 + \frac{e^{-a x}}{a^2 + 4} \mathrm{Sin}\,x + \frac{2 e^{-a x}}{a (a^2 + 4)} \mathrm{Cos}\,x \quad (101)$$

We now compute the limit as a→0 by writing the first few terms of a Laurent series about a=0:

$$y = A\,\mathrm{Sin}\,x + B\,\mathrm{Cos}\,x - 2 + x^2 + \frac{\mathrm{Cos}\,x}{2 a} - \frac{1}{2} x\,\mathrm{Cos}\,x + \frac{\mathrm{Sin}\,x}{4} + O(a) \quad (102)$$

If we rearrange terms we have

$$y \to \left(A + \frac{1}{4}\right) \mathrm{Sin}\,x + \left(B + \frac{1}{2 a}\right) \mathrm{Cos}\,x - 2 + x^2 - \frac{1}{2} x\,\mathrm{Cos}\,x \quad (103)$$

Notice that the terms in parenthesis are just constants, as such we can simply define two new constants C and D:

$$y \to C\,\mathrm{Sin}\,x + D\,\mathrm{Cos}\,x - 2 + x^2 - \frac{1}{2} x\,\mathrm{Cos}\,x \quad (104)$$

Notice that, up to a choice of arbitrary constants, this is identical to the solution found in part b.

d)

If we set a=0, our solution from part a is:

$$y = A\,\mathrm{Sin}\,x + B\,\mathrm{Cos}\,x - 2 + x^2 + \frac{1}{1 - \omega^2} \mathrm{Sin}\,\omega\,x \quad (105)$$

Taylor expanding about ω=1 gives:

$$y = \left(A - \frac{1}{2(\omega - 1)} + \frac{1}{4}\right) \mathrm{Sin}\,x + B\,\mathrm{Cos}\,x - 2 + x^2 - \frac{1}{2} x\,\mathrm{Cos}\,x + O(\omega - 1) \quad (106)$$

As in part (c) we may redefine our arbitrary constant so as to absorb the singular term and 1/4. Letting $\omega \to 1$ gives:

$$y \to C\,\mathrm{Sin}\,x + B\,\mathrm{Cos}\,x - 2 + x^2 - \frac{1}{2} x\,\mathrm{Cos}\,x \quad (107)$$

Notice that, up to a choice of arbitrary constants, this is identical to the solution found in part b.

Problem 6 (2×5 points)

Identify all the singular points of the following differential equations, and classify them [as regular singular points or essential singularities]. Be sure to consider the point at infinity (i.e. investigate the behavior of the equations with the substitution z=1/x as z=1/x→0).

a)

$$(1 - x^2) y'' - x y' + \alpha^2 y = 0 \quad \text{(Chebyshev)} \quad (108)$$

b)

$$x^2 y'' + x y' + (x^2 - \alpha^2) y = 0 \quad \text{(Bessel)} \quad (109)$$

Solution to Problem 6

In general, a second order homogeneous linear ODE written in the form

$$y'' + p(x)\,y' + q(x)\,y = 0 \qquad)$$

Gives solutions which are continuous functions whenever p and q are continuous. When p and/or q have singularities we say that the these singularities are regular singular points when has at most a first order pole and q has at most a second order pole. Otherwise we say that the singularities are irregular or essential. To examine the point at infinity we change variables to z=1/x

$$\begin{aligned} \frac{dy}{dx} &= \frac{-1}{x^2}\frac{dy}{dz} = -z^2\frac{dy}{dz} \\ \frac{d^2y}{dx^2} &= \frac{2}{x^3}\frac{dy}{dz} + \frac{1}{x^4}\frac{d^2y}{dz^2} = 2z^3\frac{dy}{dz} + z^4\frac{d^2y}{dz^2} \end{aligned} \tag{111}$$

This gives:

$$\begin{aligned} &\frac{d^2y}{dz^2} + P(z)\frac{dy}{dz} + Q(z)\,y = 0 \\ &P(z) = \frac{2}{z} - \frac{1}{z^2}\,p\left(\frac{1}{z}\right) \\ &Q(z) = \frac{1}{z^4}\,q\left(\frac{1}{z}\right) \end{aligned} \tag{112}$$

a)

$$(1 - x^2)\,y'' - x\,y' + \alpha^2\,y = 0 \tag{113}$$

Rewriting this as:

$$y'' - \frac{x}{1 - x^2}\,y' + \frac{\alpha^2}{1 - x^2}\,y = 0 \tag{114}$$

Shows that both p and q have simple poles at ±1. Hence these points are regular singular points. By transforming the independent variable, z=1/x, we examine z=0 to find what type of points x=±∞ are. We have

$$\frac{d^2y}{dz^2} + \frac{2z^2 - 1}{z(z^2 - 1)}\frac{dy}{dz} + \frac{\alpha^2}{z^2(z^2 - 1)}\,y = 0 \tag{115}$$

Since P(z) has a first order pole and Q(z) a second order pole at z=0, the point at infinity is a regular singular point for the original ODE

b)

$$x^2\,y'' + x\,y' + (x^2 - \alpha^2)\,y = 0 \tag{116}$$

First write this as:

$$y'' + \frac{1}{x}\,y' + \left(1 - \frac{\alpha^2}{x^2}\right)y = 0 \tag{117}$$

Since p(x) has a simple pole and q(x) a second order pole at x=0, this point is a regular singular point. Changing variables to z=1/x gives:

$$\frac{d^2y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + \left(\frac{1}{z^4} - \frac{\alpha^2}{z^2}\right)y = 0 \tag{118}$$

Since Q(z) has a fourth order pole at z=0, the point at infinity is an essential singularity of the original ODE.

Problem 7 (4×7 points)

Method of Frobenius

a) Find the two linearly independent series solutions about x=0 to the Chebyshev differential equation

$$(1 - x^2)\, y'' - x\, y' + \alpha^2\, y = 0 \qquad)$$

You should check the linear independence by checking that the Wronskian of (the first two terms of) your two series is non-zero for small x.

b) What is the radius of convergence of your series in (a)? Could you have anticipated this by considering the form of the equation?

c) Substitute z=1/x in the Chebyshev differential equation, and find the series solution for the resulting equation about z=0 (i.e. about x=∞). There are several special cases, depending on the values of the parameter α. To avoid doing too much tedious algebra, you may restrict your answer to consideration of three cases: $\alpha=0$, $\alpha=2$, and $\alpha\neq k/2$ where k is an integer. For 7 points extra credit, you may consider the remaining cases (α equal to other integers and half integers)

d) What is the radius of convergence of your series in (c)? Could you have anticipated this by considering the form of the equation and the theorems given in class?

Solution to Problem 7

a)

$$(1 - x^2)\, y'' - x\, y' + \alpha^2\, y = 0 \qquad (120)$$

This can be solved on the interval (-1,1) to get an exact solution (using Mathematica for example)

$$y = A \operatorname{Cosh}(\alpha \operatorname{ArcSin} x) + B \operatorname{Sinh}(\alpha \operatorname{ArcSin} x) \qquad (121)$$

Ignoring this, we'll try to find two linearly independent series solutions without using this formula. In problem 6 we found the ODE to have regular singular points at x=±1 and ∞. Since x=0 is not a singular point, but rather an ordinary point. So we look for a solution of the form:

$$y = \sum_{n=0}^{\infty} a_n\, x^n \qquad (122)$$

Computing derivatives and plugging into the ODE gives:

$$(1 - x^2) \sum_{n=2}^{\infty} n\,(n-1)\, a_n\, x^{n-2} - x \sum_{n=1}^{\infty} n\, a_n\, x^{n-1} + \alpha^2 \sum_{n=0}^{\infty} a_n\, x^n = 0 \qquad (123)$$

By expanding the first term we find:

$$\sum_{n=2}^{\infty} n\,(n-1)\, a_n\, x^{n-2} - \sum_{n=1}^{\infty} n\, a_n\, x^n + \alpha^2 \sum_{n=0}^{\infty} a_n\, x^n - \sum_{n=2}^{\infty} n\,(n-1)\, a_n\, x^n = 0 \qquad (124)$$

In order for these to be combined into a single power series, we change the index in the first series:

$$\sum_{n=0}^{\infty} (n+2)\,(n+1)\, a_{n+2}\, x^n - \sum_{n=1}^{\infty} n\, a_n\, x^n + \alpha^2 \sum_{n=0}^{\infty} a_n\, x^n - \sum_{n=2}^{\infty} n\,(n-1)\, a_n\, x^n = 0 \qquad (125)$$

We then remove the first few terms of some of the series, and collect what remains into a single series:

$$(2\, a_2 + \alpha^2\, a_0) + (6\, a_3 + (\alpha^2 - 1)\, a_1)\, x + \sum_{n=2}^{\infty} ((n+2)\,(n+1)\, a_{n+2} + (\alpha^2 - n^2)\, a_n)\, x^n = 0 \qquad (126)$$

Since any two convergent power series that are equal through out their common domain of convergence must, in fact, be the same series, we know that the only way for this expression (assumed to be convergent) to be zero is if each of the coefficients is zero:

$$a_2 = -\frac{\alpha^2}{2} a_0$$
$$a_3 = \frac{1-\alpha^2}{6} a_1 \qquad (127)$$
$$a_{n+2} = \frac{n^2-\alpha^2}{(n+2)(n+1)} a_n$$

Since the odd and even terms are coupled separately, we may separate them into two series, with a separate recurrence relation for the coefficients of each.

$$y = A\sum_{n=0}^{\infty} b_n x^{2n} + B\sum_{n=0}^{\infty} c_n x^{2n+1}$$
$$b_0 = 1$$
$$b_{n+1} = \frac{4n^2-\alpha^2}{(2n+2)(2n+1)} b_n \qquad (128)$$
$$c_0 = 1$$
$$c_{n+1} = \frac{(2n+1)^2-\alpha^2}{(2n+3)(2n+2)} c_n$$

For our two series solutions, the first 2 terms of the Wronskian are given by:

$$W = \left(1-\frac{\alpha^2}{2}x^2+..\right)'\left(x+\frac{1-\alpha^2}{6}x^3+..\right) - \left(1-\frac{\alpha^2}{2}x^2+..\right)\left(x+\frac{1-\alpha^2}{6}x^3+..\right)' =$$
$$1+\frac{1-4\alpha^2}{2}x^2+\ldots \qquad (129)$$

Since W(0)≠0, by Abel's theorem given in class the W(x)≠0 for all x in some interval about 0. Hence these solutions are linearly independent in some neighborhood of the origin.

b)

We check to see what the radius of convergence of each of these series is. Applying the ratio test, the radii of convergence for each series is given by:

$$\lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)(2n+1)}{4n^2-\alpha^2}\right| = 1$$
$$\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(2n+3)(2n+2)}{(2n+1)^2-\alpha^2}\right| = 1 \qquad (130)$$

So both series converge in $x\in(-1,1)$. Since the ODE has regular singular points at $x=\pm 1$, it is not surprising that the largest interval around x=0 on which these solution series converge is (-1,1). In fact, by the theorem discussed in class, the radius of convergence should be at least 1.

c)

In problem 6 we showed that the change of variables z=1/x gives the ODE

$$(z^2-1)\frac{d^2y}{dz^2}+\left(2z-\frac{1}{z}\right)\frac{dy}{dz}+\frac{\alpha^2}{z^2}y = 0 \qquad (131)$$

This can be solved (using Mathematica for example) to find

$$y = A\left(\frac{z}{1+\sqrt{1-z^2}}\right)^{\alpha} + B\left(\frac{z}{1+\sqrt{1-z^2}}\right)^{-\alpha} \qquad (132)$$

Ignoring this convenience, we proceed to find a series solution. The ODE has a regular singular point at z=0, so we look for a solution of the form:

$$y = \sum_{n=0}^{\infty} a_n z^{n+\nu} \tag{133}$$

We substitute this into the equation to find:

$$(z^2 - 1) \sum_{n=0}^{\infty} (n+\nu)(n+\nu-1) a_n z^{n+\nu-2} + \left(2z - \frac{1}{z}\right) \sum_{n=0}^{\infty} (n+\nu) a_n z^{n+\nu-1} + \frac{\alpha^2}{z^2} \sum_{n=0}^{\infty} a_n z^{n+\nu} = 0 \tag{134}$$

Expanding and simplifying gives:

$$\sum_{n=0}^{\infty} (n+\nu)(n+\nu+1) a_n z^{n+\nu} + \sum_{n=0}^{\infty} (\alpha^2 - (n+\nu)^2) a_n z^{n+\nu-2} = 0 \tag{135}$$

We change the index in the last series and combine things into a single sum with a few extra terms left over:

$$\begin{aligned}&(\alpha^2 - \nu^2) a_0 z^{\nu-2} + (\alpha^2 - (1+\nu)^2) a_1 z^{\nu-1} + \\ &\quad \sum_{n=0}^{\infty} ((n+\nu)(n+\nu+1) a_n + (\alpha^2 - (n+2+\nu)^2) a_{n+2}) z^{n+\nu} = 0\end{aligned} \tag{136}$$

For the same reasons used in part (a) we set the coefficients to zero:

$$\begin{aligned}&(\alpha^2 - \nu^2) a_0 = 0 \\ &(\alpha - 1 - \nu)(\alpha + 1 + \nu) a_1 = 0 \\ &a_{n+2} = \frac{(n+\nu)(n+\nu+1)}{(n+2+\nu)^2 - \alpha^2} a_n\end{aligned} \tag{137}$$

We observe that the even and odd terms are decoupled from one another. Implicit in our ansatz is that $a_0 \neq 0$. This requirement shows that

$$\nu_\pm = \pm\alpha \tag{138}$$

There are now several cases, depending on the value of α.

(i) 2α is not an integer

When 2α is not an integer, the two roots of the indicial equation are distinct and don't differ by an integer. From the class notes we know that for this case two linearly independent solutions are given by:

$$\begin{aligned}&a_1 = 0 \\ &a_{n+2} = \frac{(n+\nu_\pm)(n+\nu_\pm+1)}{(n+2+\nu_\pm)^2 - \alpha^2} a_n\end{aligned} \tag{139}$$

So our general solution is:

$$\begin{aligned}&y = A \sum_{n=0}^{\infty} b_n z^{2n+\alpha} + B \sum_{n=0}^{\infty} c_n z^{2n-\alpha} \\ &b_0 = c_0 = 1 \\ &b_{n+1} = \frac{(2n+\alpha)(2n+\alpha+1)}{(2n+2+\alpha)^2 - \alpha^2} b_n \\ &c_{n+1} = \frac{(2n-\alpha)(2n-\alpha+1)}{(2n+2-\alpha)^2 - \alpha^2} c_n\end{aligned} \tag{140}$$

Write ou the first few terms of each:

$$y = A x^{\alpha} \left(1 + \frac{\alpha}{4} x^2 + ..\right) + B x^{-\alpha} \left(1 - \frac{\alpha}{4} x^2 + ..\right) \tag{141}$$

Check the Wronskian

$$\left(-\alpha x^{-\alpha-1} - \frac{\alpha(2-\alpha)}{4} x^{1-\alpha} + \ldots\right)\left(x^{\alpha} + \frac{\alpha}{4} x^{2+\alpha} + ..\right) - \left(x^{-\alpha} - \frac{\alpha}{4} x^{2-\alpha} + \ldots\right)\left(\alpha x^{\alpha-1} + \frac{\alpha(2+\alpha)}{4} x^{1+\alpha} + ..\right) = -2\alpha x^{-1} + O(x) \quad (142)$$

So the Wronskain is non-zero and our two solutions are linearly independent

(ii) α=0

In this case, ν=0 is a repeated root of the indicial equation so we have only produced one solution:

$$\begin{aligned} a_1 &= 0 \\ a_{n+2} &= \frac{n(n+1)}{(n+2)^2} a_n \\ y_1 &= \sum_{n=0}^{\infty} a_n z^n \end{aligned} \quad (143)$$

Which you can see is just the constant solution since the recursion relation shows all the other coefficients are zero. To find the general solution (and hence a second linearly independent solution) it is easiest to go back to the ODE

$$(z^2-1)\frac{d^2 y}{dz^2} + \left(2z - \frac{1}{z}\right)\frac{dy}{dz} \quad (144)$$

This is simply a first order ODE for y' with a regular singular point at z=0.

We let

$$y' = \sum_{n=0}^{\infty} c_n z^{n+\mu} \quad (145)$$

Plug this in and simplify:

$$-c_0(1+\mu)z^{\mu-1} - c_1(2+\mu)z^{\mu} + \sum_{n=0}^{\infty}(c_n(n+2+\mu) - c_{n+2}(n+3+\mu))z^{n+\mu} = 0 \quad (146)$$

Setting the coefficients to zero, we have

$$\begin{aligned} \mu &= -1 \\ c_1 &= 0 \\ c_{n+2} &= \frac{n+1}{n+2} c_n \end{aligned} \quad (147)$$

So we find:

$$y' = \frac{c_0}{z} + \sum_{n=0}^{\infty} c_{n+1} z^n \quad (148)$$

Integrating this gives the general solution:

$$\begin{aligned} c_1 &= 0 \\ c_{n+2} &= \frac{n+1}{n+2} c_n \\ y &= A + c_0 \ln|z| + \sum_{n=0}^{\infty} \frac{c_{n+1}}{n+1} z^{n+1} \end{aligned} \quad (149)$$

We may rewrite this to explicitly exclude the odd terms (which are all zero)

$$d_{n+1} = \frac{2n+1}{2n+2} d_n$$
$$y = A + d_0 \ln|z| + \sum_{n=1}^{\infty} \frac{d_n}{2n} z^{2n} \quad (150)$$

(iii) α=2

One solution is given by setting $\nu=\alpha=2$

$$\begin{aligned} a_1 &= 0 \\ a_{n+2} &= \frac{n+3}{n+6} a_n \\ y_1 &= \sum_{n=0}^{\infty} a_n z^{n+2} \end{aligned} \quad (151)$$

We may write this in a way to explicitly exclude the odd indexed coefficients (which are all 0)

$$\begin{aligned} b_0 &= 1 \\ b_{n+1} &= \frac{2n+3}{2n+6} b_n \\ y_1 &= \sum_{n=0}^{\infty} b_n z^{2n+2} \end{aligned} \quad (152)$$

The first few terms are:

$$x^2 + \frac{1}{2} x^4 + \frac{5}{16} x^6 + .. \quad (153)$$

Since 2α is an integer, we see that the two roots of the indicial equation are distinct, but differ by an integer. The class notes tell us that our other Frobenius solution may not be linearly independent of this one. Let us set $\nu=-\alpha$ and find this other solution:

The recusion relation gives:

$$\begin{aligned} a_1 &= 0 \\ a_{n+2} &= \frac{n-1}{n+2} a_n \text{ for } n \neq 2 \end{aligned} \quad (154)$$

So the first few terms in this solution are

$$a_0 \left(x^{-2} - \frac{1}{2}\right) + a_4 \left(x^2 + \frac{1}{2} x^4 + \frac{5}{16} x^6\right) \quad (155)$$

You can see that the second term in parenthesis is just the solution we already found. We check to see if the first term in parenthesis provides a second linearly independent solution by checking the Wronskian:

$$(-2x^{-3}) \left(x^2 + \frac{1}{2} x^4 + ..\right) - \left(x^{-2} - \frac{1}{2}\right) (2x + 2x^3 + ..) = -2x^{-1} + O(x) \quad (156)$$

So the Wronskian is non-zero, and thus our general solution is:

$$y = A\left(x^{-2} - \frac{1}{2} + ..\right) + B\left(x^2 + \frac{1}{2} x^4 + \frac{5}{16} x^6 + ..\right) \quad (157)$$

Bonus The case $2\alpha \in \mathbb{Z}^+$ in general

From above, when we set $\nu=\alpha$ we get one solution:

$$y_1 = x^{\alpha} \left(1 + \frac{\alpha}{4} x^2 + \frac{\alpha(\alpha+3)}{32} x^4 + ..\right) \quad (158)$$

When we set ν=-α the equations for the coefficents become:

$$(2\,\alpha - 1)\,a_1 = 0 \\ (n+2)\,(n+2-2\,\alpha)\,a_{n+2} = (n-\alpha)\,(n-\alpha+1)\,a_n \tag{159}$$

Observe that α=1/2 would cause this first equation to be satisfied automatically. So we take this case separately.

(i) α=1/2

The recursion relation is

$$a_{n+2} = \frac{n^2 - 1/4}{(n+2)\,(n+1)}\,a_n \tag{160}$$

So we write out the first few terms of the solution

$$y_2 = x^{-1/2}\,a_0\left(1 - \frac{1}{8}\,x^2 + ..\right) + x^{1/2}\,a_1\left(1 + \frac{1}{8}\,x^2 + ..\right) \tag{161}$$

The solution we found for ν=α=1/2 was

$$y_1 = x^{1/2}\left(1 + \frac{1}{8}\,x^2 + ..\right) \tag{162}$$

Notice that this is the same as second series in our y_2 solution. Let's check that the remaining part of our y_2 solution is linearly independent of y_1 by computing the Wronskian:

$$\left(\frac{-1}{2}\,x^{-3/2} - \frac{3}{16}\,x^{1/2} + ...\right)\left(x^{1/2} + \frac{1}{8}\,x^{5/2} + ..\right) - \\ \left(x^{-1/2} - \frac{1}{8}\,x^{3/2} + ...\right)\left(\frac{1}{2}\,x^{-1/2} + \frac{5}{16}\,x^{3/2} + ..\right) = -x^{-1} + O(x) \tag{163}$$

Since the Wronskain is non-zero, the solutions are independent. Hence our general solution is:

$$y = A\,x^{-1/2}\left(1 - \frac{1}{8}\,x^2 + ..\right) + B\,x^{1/2}\left(1 + \frac{1}{8}\,x^2 + ..\right) \tag{164}$$

Now consider α≠1/2. Recall that he equations for the coefficents are:

$$a_1 = 0 \\ (n+2)\,(n+2-2\,\alpha)\,a_{n+2} = (n-\alpha)\,(n-\alpha+1)\,a_n \tag{165}$$

Notice that the term on the left side will vanish when n=2α-2. So the recusive relation we write is of the form:

$$a_1 = 0 \\ (\alpha - 2)\,(\alpha - 1)\,a_{2\,\alpha-2} = 0 \\ a_{n+2} = \frac{(n-\alpha)\,(n-\alpha+1)}{(n+2)\,(n+2-2\,\alpha)}\,a_n \;\text{ for } n \neq 2\,\alpha - 2 \tag{166}$$

From the second equation, α=2 and α=1 are obviously special cases. Since α=2 has been treated above, we now discuss the α=1 case.

(ii) α=1

The recusive relation becomes:

$$a_1 = 0 \\ a_{n+2} = \frac{n-1}{n+2}\,a_n \;\text{ for } n \neq 0 \tag{167}$$

The first few terms of the solution are:

$$y_2 = a_0\,x^{-1} + a_2\,x\left(1 + \frac{1}{4}\,x^2 + \frac{1}{8}\,x^4 + ..\right) \tag{168}$$

The solution we already found for $\nu=\alpha=1$ is:

$$y_1 = x\left(1 + \frac{1}{4} x^2 + \frac{1}{8} x^4 + ..\right) \tag{169}$$

Notice that this is identical to the second part of our y_2 solution. We now check to see if the remaining part of our y_2 solution is linearly independent of y_1 by checking the Wronskian:

$$-x^{-2}\left(x + \frac{1}{4} x^3 + ..\right) - x^{-1}\left(1 + \frac{3}{4} x^2 + ..\right) = -2\, x^{-1} + O(x) \tag{170}$$

Since the Wronskian is non-zero, our solutions are independent. So the general solution for $\alpha=1$ is

$$y = Ax^{-1} + B\, x\left(1 + \frac{1}{4} x^2 + \frac{1}{8} x^4 + ..\right) \tag{171}$$

(iii) $\alpha\neq 1/2,1,2$

The recursive relation is:

$$\begin{aligned} a_1 &= 0 \\ a_{2\alpha-2} &= 0 \\ a_{n+2} &= \frac{(n-\alpha)(n-\alpha+1)}{(n+2)(n+2-2\alpha)}\, a_n \ \text{ for } n \neq 2\alpha - 2 \end{aligned} \tag{172}$$

Since n=2α-2 can not be inserted into the recusion relation, the coefficient $a_{2\alpha}$ can not be written in terms of previous coefficients. Hence the solution will be of the form:

if 2α is odd

$$\begin{aligned} y_2 = {} & x^{-\alpha}\,(a_0 + a_2\, x^2 + a_4\, x^4 + ... + a_{2\alpha-1}\, x^{2\alpha-1} + a_{2\alpha+1}\, x^{2\alpha+1} + ..) + \\ & x^{-\alpha}\,(a_{2\alpha}\, x^{2\alpha} + a_{2\alpha+2}\, x^{2\alpha+2} + a_{2\alpha+4}\, x^{2\alpha+4} + ..) \end{aligned} \tag{173}$$

if 2α is even

$$\begin{aligned} y_2 = {} & x^{-\alpha}\,(a_0 + a_2\, x^2 + a_4\, x^4 + ... + a_{2\alpha-4}\, x^{2\alpha-4} + a_{2\alpha-2}\, x^{2\alpha-2}) + \\ & x^{-\alpha}\,(a_{2\alpha}\, x^{2\alpha} + a_{2\alpha+2}\, x^{2\alpha+2} + a_{2\alpha+4}\, x^{2\alpha+4} + ..) \end{aligned} \tag{174}$$

Where all the coefficients of terms in parenthesis only depend on the leading coefficient, that is we s could write:

if 2α is odd

$$\begin{aligned} y_2 = {} & a_0\, x^{-\alpha}\,(1 + b_2\, x^2 + b_4\, x^4 + ... + b_{2\alpha-1}\, x^{2\alpha-1} + b_{2\alpha+1}\, x^{2\alpha+1} + ..) + \\ & a_{2\alpha}\, x^{-\alpha}\,(x^{2\alpha} + b_{2\alpha+2}\, x^{2\alpha+2} + b_{2\alpha+4}\, x^{2\alpha+4} + ..) \end{aligned} \tag{175}$$

if 2α is even

$$\begin{aligned} y_2 = {} & a_0\, x^{-\alpha}\,(1 + b_2\, x^2 + b_4\, x^4 + ... + b_{2\alpha-4}\, x^{2\alpha-4} + b_{2\alpha-2}\, x^{2\alpha-2}) + \\ & a_{2\alpha}\, x^{-\alpha}\,(x^{2\alpha} + b_{2\alpha+2}\, x^{2\alpha+2} + b_{2\alpha+4}\, x^{2\alpha+4} + ..) \end{aligned} \tag{176}$$

Where all of the b_n's are just constants having no dependence on the a's and depending only on the value of n. To make this clear, we write out some terms:

$$a_{n+2} = \frac{(n-\alpha)(n-\alpha+1)}{(n+2)(n+2-2\alpha)}\, a_n \ \text{ for } n \neq 2\alpha - 2 \tag{177}$$

if 2α is odd

$$y_2 = a_0\, x^{-\alpha}\left(1 - \frac{\alpha}{4} x^2 + \frac{\alpha(\alpha-3)}{32} x^4 + ...\right) + a_{2\alpha}\, x^{\alpha}\left(1 + \frac{\alpha}{4} x^2 + \frac{\alpha(\alpha+3)}{32} x^4 + ..\right) \tag{178}$$

if 2α is even

$$y_2 = a_0\, x^{-\alpha} \left(1 - \frac{\alpha}{4}\, x^2 + \frac{\alpha\,(\alpha - 3)}{32}\, x^4 + \ldots + O\left(x^{2\,\alpha - 2}\right)\right) + a_{2\,\alpha}\, x^{\alpha} \left(1 + \frac{\alpha}{4}\, x^2 + \frac{\alpha\,(\alpha + 3)}{32}\, x^4 + ..\right) \qquad (17\ldots)$$

Compare these now to the previous solution we found:

$$y_1 = x^{\alpha} \left(1 + \frac{\alpha}{4}\, x^2 + \frac{\alpha\,(\alpha + 3)}{32}\, x^4 + ..\right) \qquad (180)$$

Notice that for 2α even or odd, the second part of the y_2 solution is identical to the y_1 solution. We now check that the Wronskian of the remaining parts of y_2 are linearly independt of y_1 by checking the wronskain:

$$\begin{aligned} &\left(-\alpha x^{-\alpha - 1} - \frac{\alpha\,(2 - \alpha)}{4}\, x^{1-\alpha} + \ldots\right)\left(x^{\alpha} + \frac{\alpha}{4}\, x^{2+\alpha} + ..\right) - \\ &\quad \left(x^{-\alpha} - \frac{\alpha}{4}\, x^{2-\alpha} + \ldots\right)\left(\alpha x^{\alpha - 1} + \frac{\alpha\,(2 + \alpha)}{4}\, x^{1+\alpha} + ..\right) = -2\,\alpha x^{-1} + O(x) \end{aligned} \qquad (181)$$

So, regardless of even or odd, when 2α is an integer ≥3, we have two linearly independent solutions of the form:

if 2α is odd

$$y = A\left(x^{-\alpha} - \frac{\alpha}{4}\, x^{2-\alpha} + \frac{\alpha\,(\alpha - 3)}{32}\, x^{4-\alpha} + \ldots\right) + B\left(x^{\alpha} + \frac{\alpha}{4}\, x^{2+\alpha} + \frac{\alpha\,(\alpha + 3)}{32}\, x^{4+\alpha} + ..\right) \qquad (182)$$

if 2α is even

$$y_2 = A\left(x^{-\alpha} - \frac{\alpha}{4}\, x^{2-\alpha} + \frac{\alpha\,(\alpha - 3)}{32}\, x^{4-\alpha} + \ldots + O\left(x^{\alpha - 2}\right)\right) + B\left(x^{\alpha} + \frac{\alpha}{4}\, x^{2+\alpha} + \frac{\alpha\,(\alpha + 3)}{32}\, x^{4+\alpha} + ..\right) \qquad (183)$$

Note that the only difference between these is that when 2α is even, one of the solutions is a finite series, i.e. $x^{-\alpha}$ times a polynomial of degree 2α-2.

d)

case(i): 2α not an integer

The solution we found was:

$$\begin{aligned} y &= A \sum_{n=0}^{\infty} b_n\, z^{2\,n+\alpha} + B \sum_{n=0}^{\infty} c_n\, z^{2\,n-\alpha} \\ b_0 &= c_0 = 1 \\ b_{n+1} &= \frac{(2\,n + \alpha)\,(2\,n + \alpha + 1)}{(2\,n + 2 + \alpha)^2 - \alpha^2}\, b_n \\ c_{n+1} &= \frac{(2\,n - \alpha)\,(2\,n - \alpha + 1)}{(2\,n + 2 - \alpha)^2 - \alpha^2}\, c_n \end{aligned} \qquad (184)$$

The radii of convergence of these series are given by:

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right| &= \lim_{n\to\infty}\left|\frac{(2\,n + 2 + \alpha)^2 - \alpha^2}{(2\,n + \alpha)\,(2\,n + \alpha + 1)}\right| = 1 \\ \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| &= \lim_{n\to\infty}\left|\frac{(2\,n + 2 - \alpha)^2 - \alpha^2}{(2\,n - \alpha)\,(2\,n - \alpha + 1)}\right| = 1 \end{aligned} \qquad (185)$$

Since the ODE has regular singular points at z=±1, it isn't surprising that the series doesn't converge for |z|>1. By the theorem given in class, since the distance from z=0 to the singularities at z=±1 is 1, we know the radius of convergence is at least 1.

case(ii) α=0

$$y = A + a_0 \ln|z| + \sum_{n=1}^{\infty} \frac{a_{2n}}{2n} z^{2n} \tag{186}$$

$$a_{2n+2} = \frac{2n+1}{2n+2} a_{2n}$$

The radius of convergence is given by:

$$\lim_{n\to\infty} \left| \frac{\frac{a_{2n}}{2n}}{\frac{a_{2n+2}}{2n+2}} \right| = \lim_{n\to\infty} \left| \frac{(2n+2)^2}{2n(2n+1)} \right| = 1 \tag{187}$$

By the same reason as case (i) this is what we might have expected.

case (iii): α=2

Recall that one of our solutions had coefficients given by:

$$b_{n+1} = \frac{2n+3}{2n+6} b_n \tag{188}$$

and the other solution was a finite series. Finite series converge unconditionally, so we only need to check the radius of convergence for the infintie series:

$$\lim_{n\to\infty} \left| \frac{b_n}{b_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{2n+6}{2n+3} \right| = 1 \tag{189}$$

By the same reason as case (i) this is what we might have expected.