ACM95/100b
Problem Set 1 Solutions
01/12/04

Problem 1 (2×7 points)

Solve the following linear initial value problems and in each case describe the interval on which the solution is defined

a)

$$\begin{aligned} &y' + 2\,x\,y = e^{-x^2} \\ &y(0) = -1 \end{aligned} \tag{1}$$

b)

$$\begin{aligned} &t\,y' + 2\,y = t^2 - t + 1 \\ &y(1) = 1/2 \end{aligned} \tag{2}$$

Solution to Problem 1

Note that "initial value problem" is typically meant to refer to a problem with prescribed data at $t=\tau$ with the desired solution only being examined for $t\geq\tau$. In what follows I have discussed continuity and existence for the full range $t\in(-\infty,\infty)$. If students only consider $t\in[\tau,\infty)$ they should still receive full credit.

From the class notes, an initial value problem of the form:

$$\begin{aligned} &y' + p\,y = q \\ &y(a) = b \end{aligned} \tag{3}$$

Can be solved by use of an integrating factor

$$I = e^{\int_a^x p(t)\,dt} \tag{4}$$

Multiplying by this factor gives:

$$\left(e^{\int_a^x p(t)\,dt}\,y\right)' = q\,e^{\int_a^x p(t)\,dt} \tag{5}$$

After an integration and rearrangement we have:

$$y = e^{-\int_a^x p(t)\,dt}\left(A + \int_a^x q(s)\,e^{\int_a^s p(t)\,dt}\,ds\right) \tag{6}$$

Applying the initial condition gives us the value of the arbitrary integration constant A:

$$b = A \tag{7}$$

More generally the integration factor can be chosen to be

$$I = e^{\int p(x)\,dx} \tag{8}$$

So that the most general solution to the ODE (ignoring the initial condition) is:

$$y = e^{-\int p(x)\,dx}\left(A + \int q(x)\,e^{\int p(x)\,dx}\,dx\right) \tag{9}$$

a)

$$\begin{aligned} &y' + 2\,x\,y = e^{-x^2} \\ &y(0) = -1 \end{aligned} \tag{10}$$

Applying the previously derived formula we find:

$$y = e^{-\int_0^x 2\,t\,dt}\left(-1+\int_0^x e^{-s^2}\,e^{\int_0^s 2\,t\,dt}\,ds\right) \tag{11}$$

After simplification this becomes:

$$y = e^{-x^2}\,(x-1) \tag{12}$$

Clearly the solution remains bounded for all -∞≤x<∞, and so the solution exists and is continous everywhere.

b)

The problem may be rewritten as:

$$y' + \frac{2}{t}\,y = t - 1 + \frac{1}{t}$$
$$y(1) = 1/2 \tag{13}$$

The formula then gives:

$$y = e^{-\int_1^t \frac{2}{s}\,ds}\left(\frac{1}{2} + \int_1^t \left(s - 1 + \frac{1}{s}\right) e^{\int_1^s \frac{2}{y}\,dy}\,ds\right) \tag{14}$$

Simplifying:

$$y = \frac{1}{t^2}\left(\frac{1}{2} + \int_1^t (s^3 - s^2 + s)\,ds\right) = \frac{1}{12\,t^2} + \frac{1}{2} - \frac{t}{3} + \frac{t^2}{4} \tag{15}$$

We see that the solution becomes unbounded at t=0. Hence the solution exists only for 0<t<∞.

Problem 2 (15 points)

Solve the following real-valued initial value problem [4 points]:

$$x\,y'(x) + A y(x) = 1 + x^2$$
$$y(1) = 1 \tag{16}$$

for all (positive, zero and negative) values of the constant A, and then answer the following questions: Is your solution actually valid for all values of the constant A? If you weren't careful in your initial solution, you may need to amend it to include some different functional forms for some values of A. [4 points] Over what range of x is the solution defined and continuous; your answer may depend on A [3 points]? Ignore the condition y(1)=1 and find that function y(x) which obeys the differential equation is bounded at the origin; you need not find this solution for special values of A that require different functional forms [4 points].

Solution to Problem 2

We first rewrite the equation:

$$y'(x) + \frac{A}{x}\,y(x) = \frac{1}{x} + x \tag{17}$$

and then apply the formula derived in the solution of problem 1:

$$y = e^{-\int_1^x \frac{A}{t}\,dt}\left(1 + \int_1^x \left(\frac{1}{s} + s\right) e^{\int_1^s \frac{A}{t}\,dt}\,ds\right) \tag{18}$$

Simplifying gives:

$$y = x^{-A} + x^{-A}\int_1^x (s^{A-1} + s^{A+1})\,ds \tag{19}$$

Depending on the value of A, the integral is computed differently yielding different possible solutions:

$$y = \begin{cases} \frac{1}{2} + \frac{1}{2} x^2 + \ln|x| & A = 0 \\ -\frac{1}{2} + \frac{3}{2} x^2 + x^2 \ln|x| & A = -2 \\ \frac{1}{A} + \frac{1}{A+2} x^2 & A = \pm\sqrt{2} \\ \frac{1}{A} + \frac{1}{A+2} x^2 + \frac{A^2-2}{A(A+2)} x^{-A} & \text{otherwise} \end{cases} \tag{20}$$

The solution takes on a special form for A=-2 and A=0.
For A=0 the solution is defined and continuous only for x>0.
For A=-2 the solution is defined for all x except x=0, and since

$$\lim_{x\to 0^+} x^2 \ln|x| = \lim_{x\to 0^-} x^2 \ln|x| = 0 \tag{21}$$

the solution is continuous at x=0 only if we define y(0)=-1/2.
If A is any negative integer except -2 the solution is continuous for $-\infty<x<\infty$.
For all positive values of A, except A=$\sqrt{2}$, the term x^{-A} is undefined at x=0 and hence the solution is continuous only for $0<x<\infty$.
For A= $\pm\sqrt{2}$ the solution is continuous everywhere.
For some negative non-integer values of A, the solution is bounded and continuous for $0<x<\infty$ and for others the solution is bounded and continuous on $-\infty<x<\infty$. For example, if A=-1/2 then x^{-A} is bounded at x=0 but not continuous since it is complex for x<0. But if A=-1/3 then x^{-A} is both bounded and continuous at x=0. In general, if A is a negative irrational number, except -$\sqrt{2}$, or if A=-p/q with p odd and q even then the solution is discontinuous at x=0. If A=-p/q with q odd then the solution is continuous at x=0.

Recall that the solution to the ODE y'+py=q is, in general:

$$y = e^{-\int p(x)\,dx} \left(C + \int q(x)\, e^{\int p(x)\,dx}\, dx\right) \tag{22}$$

In our case, ignoring A=0 and A=-2, this gives:

$$y = C x^{-A} + x^{-A} \int (x^{A-1} + x^{A+1})\, dx = \frac{1}{A} + \frac{x^2}{2+A} + C x^{-A} \tag{23}$$

For C≠0 this is only a bounded continuous function at the origin if A is a negative integer or if A is a negative rational number -p/q with q odd. If A is to be chosen arbitrarily it is necessary to set C=0 to achieve a continuous solution at x=0.

Problem 3 (10 points)

Consider the same DE as in problem 2 complexified, i.e. with real x replaced by complex z and real y(x) replaced with complex w(z). Assume w(1+0i)=1+0i. Find an analytic solution w(z) for general real A (you may ignore any 'peculiar values' of A for purposes of this problem), and discuss the region of the complex plane over which it is valid (5 points). Be sure to define the locations of any branch cuts you introduce (2 points). Under what circumstance can you analytically continue the solution over the whole real line x=x+0i? To the whole real line excluding one point? Compare to your answers for problem 2 (3 points).

Solution to Problem 3

$$w'(z) + \frac{A}{z} w(z) = \frac{1}{z} + z$$
$$w(1) = 1 \tag{24}$$

In general, an integrating factor for an equation of the following form:

$$y' + f(x)\, y = g(x) \tag{25}$$

Is given by

$$I = e^{\int f(x)\,dx} \tag{26}$$

This allows the ODE to be rewritten in a form easy to integrate:

$$\left(e^{\int f(x)\,dx}\, y\right)' = g(x)\, e^{\int f(x)\,dx} \tag{27}$$

In the present case, we use this formula to find the following integrating factor:

$$I = e^{\int \frac{A}{z}\,dz} \tag{28}$$

Where the integral is computed over some contour Γ connecting 1 to z and not passing through the origin. Since the integrand is analytic everywhere on Γ, the antiderivative exists:

$$\int_\Gamma \frac{A}{y}\,dy = A\,(\mathrm{Log}\, z + 2\, n\pi i) \tag{29}$$

Where Log z is some branch of the log function. This gives:

$$I = e^{\int \frac{A}{y}\,dy} = e^{A\,(\mathrm{Log}\, z + 2\, n\pi i)} = z^A\, e^{2\pi n A i} \tag{30}$$

Using this integrating factor gives:

$$(z^A\, w(z))' = z^{A-1} + z^{A+1} \tag{31}$$

Integrating along Γ gives:

$$\int_\Gamma (y^A\, w(y))'\,dy = \int_\Gamma (y^{A-1} + y^{A+1})\,dy \tag{32}$$

We assume that w is analytic on Γ and consider 3 possible cases:

(i) A is a positive integer

In this case both integrands are analytic functions and hence the integrals can be evaluated in a straightforward manor using antiderivatives

$$z^A\, w(z) - w(1) = \frac{1}{A}\, z^A + \frac{1}{A+2}\, z^{A+2} - \frac{1}{A} - \frac{1}{A+2} \tag{33}$$

Simplifying gives:

$$w(z) = \frac{1}{A} + \frac{1}{A+2}\, z^2 + \frac{A^2-2}{A\,(2+A)}\, z^{-A} \tag{34}$$

This solution is analytic for all z≠0 and hence for all real x≠0. For real z this solution is identical to that computed in problem 2.

(ii) A is a negative integer

In this case both integrands have poles at z=0 but since the contour Γ was chosen to avoid z=0 these integrands are analytic on Γ and hence the solution is the same as case (i).

$$w(z) = \frac{1}{A} + \frac{1}{A+2}\, z^2 + \frac{A^2-2}{A\,(2+A)}\, z^{-A} \tag{35}$$

Since A<0 this solution is entire moreover it exists on the whole real axis. This is exactly what we found in problem 2.

(iii) A is not an integer

In this case the origin and the point at infinity are both branch point singularities of the integrands. Let ψ be any simple curve in the complex plane connecting z=0 to $z=z_\infty$. This ψ is a suitable branch cut. If we then choose z^A to be any particular branch, then the integrands are analytic everywhere on Γ provided that Γ not intersect ψ. If we want our solution to be valid for all points on the real axis (except z=0) then we should not place branch cuts on either the positive or negative real axis, as you may have done many times before. A cut at any angle other than 0 or π will suffice. To be specific let the branch cut ψ be the negative imaginary axis and choose the following branch of z^A:

$$z^A = e^{A(\ln|z| + i\arg(z))}$$
$$\frac{-\pi}{2} \le \arg(z) < \frac{3\pi}{2} \qquad (36$$

Then if we wish to define w(z) as an analytic function for all real z except z=0 we simply integrate both integrals on some curve Γ from 1 to z not intersecting the branch cut on the negative imaginary axis. In problem 2 it wasn't possible to define the solution along the negative real axis because restricting Γ to the x axis made it impossible to avoid the branch point singularity at z=0. In the present case we have shown that w(z) can be define analytically for z=x<0 since Γ connecting 1 to z can be chosen to avoid the branch point.

Problem 4 (15 points)

For any real number a, find y(x) such that

$$y'(x) + a\,y(x) = e^{-x}$$
$$y(0) = 1 \qquad (37)$$

over the range $0 \le x < \infty$ [8 points]. Sketch a graph of y(x) for each of several values of a. Include particularly a=-100,-1,1,100, 10^4 [7 points]

Solution to Problem 4

In general, an integrating factor for an equation of the following form:

$$y' + f(x)\,y = g(x) \qquad (38)$$

Is given by

$$I = e^{\int f(x)\,dx} \qquad (39)$$

This allows the ODE to be rewritten in a form easy to integrate:

$$\left(e^{\int f(x)\,dx}\, y\right)' = g(x)\, e^{\int f(x)\,dx} \qquad (40)$$

For this problem the formula produces the integrating factor $I=e^{ax}$. This gives:

$$(e^{a x}\, y)' = e^{(a-1)\,x} \qquad (41)$$

Integrating and simplifying gives:

$$y = \begin{cases} A\, e^{-a x} + \frac{e^{-x}}{a-1} & a \ne 1 \\ (B + x)\, e^{-x} & a = 1 \end{cases} \qquad (42)$$

Implementing the initial condition gives:

$$y = \begin{cases} \frac{a-2}{a-1}\, e^{-a x} + \frac{e^{-x}}{a-1} & a \ne 1 \\ (1 + x)\, e^{-x} & a = 1 \end{cases} \qquad (43)$$

Some plots are shown below, note the different scales.

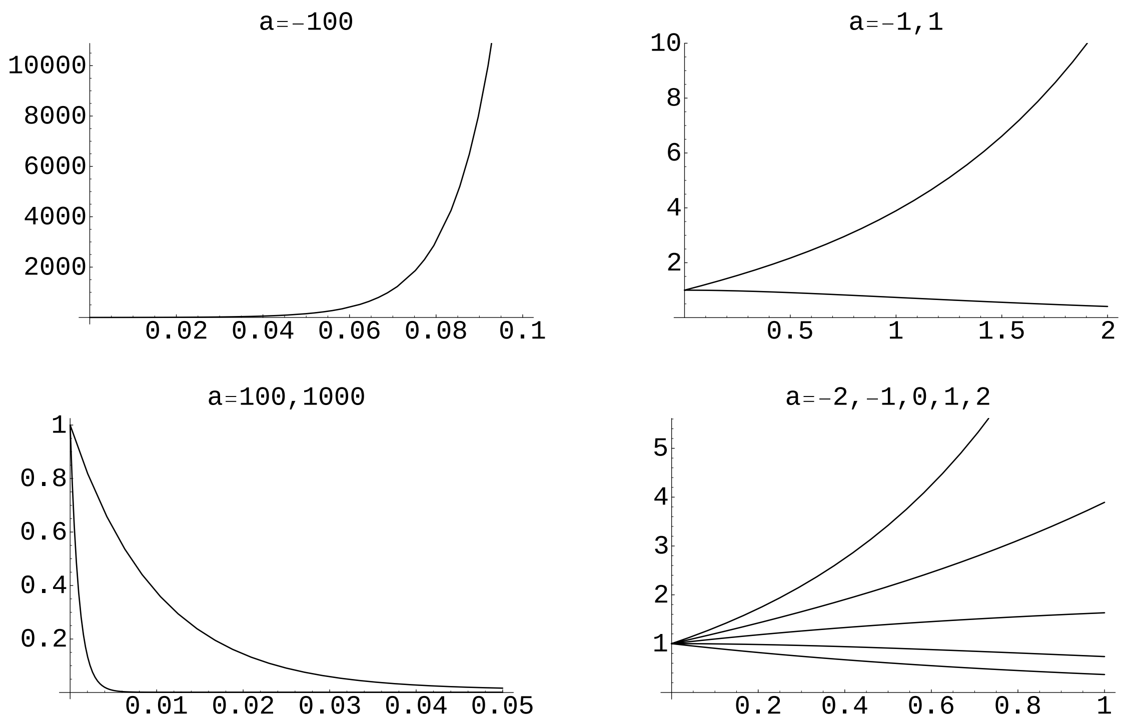

Figure 1

Problem 5 (7 points)

Consider the DE

$$y' + y/x = 1 \qquad y(1) = 1/2 \tag{44}$$

Find the vertices of the Euler polygonal approximations (described in class) to the solution, as a function of x-increment h [5 points]. Show explicitly that the approximate Euler solutions approach the actual solution as $h \to 0$ [2 points].

Solution to Problem 5

Define

$$x_n = 1 + n\,h \tag{45}$$

The Euler iterates are then given by:

$$Y_{n+1} = Y_n + h\,(1 - Y_n/x_n) = Y_n + h\left(1 - \frac{Y_n}{1 + n\,h}\right) \qquad Y_0 = 1/2 \tag{46}$$

By inspection, a solution to this initial value linear difference equation is:

$$Y_n = \frac{1 + nh}{2} \tag{47}$$

By the uniqueness of solutions to first order linear difference equations, this is the only solution.
The exact solution to the initial value problem can be obtained either by appealing to the formula given in problem 1 or by use of an integration factor. Either way we find:

$$y = x/2 \tag{48}$$

The exact solution evaluated on the x grid points is then:

$$y_n = \frac{x_n}{2} = \frac{1 + n\,h}{2} \tag{49}$$

Notice that $Y_n = y_n$ hence the Euler approximation gives the exact solution for any value of h.

Problem 6 (4x5 points)

a) A cylindrical bucket has a hole in the bottom, and is observed to be empty at time τ=0. The differential equation governing the height z(τ) of water in the bucket as a function of (appropriately scaled) time is the following 'final value problem':

$$\frac{dz}{d\tau} = -|z|^{1/2} \qquad z(0) = 0 \tag{50}$$

Notice the following: (i) height of water is a positive number (on my rulers anyway), so only solutions with z≥0 are to be considered, (ii) z(τ)=0 is *a* solution, but perhaps not the only one. Given that z(0)=0, show mathematically that the solution is unique for τ>0, but non-unique for τ<0, and give a simple explanation in terms of what you can infer (when did it empty?) from observing an empty bucket with a puddle under it for why nonuniqueness is reasonable and physical. [hint: be careful with integration constants and matching solutions]

With your insights from part (a), now consider the following initial value problem:

$$\frac{dz}{dt} = |z|^{p/q} \qquad z(0) = 0 \tag{51}$$

where p and q are positive integers with no common factors. Notice that z(t)=0 is *a* solution, but perhaps not the only one, and that 'initial value problem' means that you are to consider only t≥0. Notice that setting τ=-t in part (a) converts the 'final value problem' to the present initial value problem, with p=1, q=2.

b) Show that there are an infinite number of solutions if p<q.

c) Show that there is a unique solution if p>q.

d) Relate your results of parts (b) and (c) to the Lipshitz condition used in the proof of the uniqueness theorem sketched in class.

Solution to Problem 6

a)

$$\frac{dz}{d\tau} = -|z|^{1/2} \qquad z(0) = 0 \tag{52}$$

Since the right hand side is non-positive, for τ>0 we have z≤0. This is easily seen as follows

$$z(\tau) = \int_0^\tau \frac{dz}{dt}\,dt = -\int_0^\tau |z|^{1/2}\,dt \le 0 \tag{53}$$

Motivated by the physical problem, z=0 corresponds to an empty bucket, hence z<0 is physically impossible. Since we know that, for τ>0, z≤0, this requires that z=0 for all τ>0. Hence the solution z(τ)=0 for τ>0 is the unique physical solution.
A similar calculation shows that, for τ<0, z≥0. This is physically permissible. However, when posed with an empty bucket we have no idea of knowing when the bucket became empty.
So for τ<0 we have the following problem:

$$\frac{dz}{d\tau} = -z^{1/2} \qquad z(0) = 0 \tag{54}$$

We may rewrite the ODE as:

$$\frac{d}{d\tau}(2\,z^{1/2}) = z^{-1/2}\,\frac{dz}{d\tau} = -1 \tag{55}$$

Integrating both sides gives:

$$2\,z^{1/2} - 2\,z_0^{\;1/2} = \tau_0 - \tau \tag{56}$$

Simplifying:

$$z(\tau) = \left(z_0^{\;1/2} + \frac{\tau_0 - \tau}{2}\right)^2 \tag{57}$$

Since $\tau_0<0$ and $z(\tau)=0$ for all $\tau>0$ we may combine this solution with the zero solution to get:

$$z(\tau) = \begin{cases} \left(\frac{\tau_0-\tau}{2}\right)^2 & \tau < \tau_0 < 0 \\ 0 & \tau > \tau_0 \end{cases} \tag{58}$$

Thus for $\tau<0$ there are infinitely many physically allowable solutions corresponding to buckets that became empty at time $\tau=\tau_0\leq 0$.

b)

$$\frac{dz}{dt} = |z|^{p/q} \qquad p < q \tag{59}$$

For parts (b) and (c) we write the equation as:

$$|z|^{-p/q}\,\frac{dz}{dt} - 1 = 0 \tag{60}$$

For $p\neq q$ we may write this as:

$$\frac{d}{dt}\left(\frac{|z|^{1-p/q}}{1-p/q}\,\mathrm{sign}(z) - t\right) = 0 \tag{61}$$

We now integrate from t_0 to t on both sides:

$$\frac{|z|^{1-p/q}}{1-p/q}\,\mathrm{sign}(z) - \frac{|z_0|^{1-p/q}}{1-p/q}\,\mathrm{sign}(z_0) = t - t_0 \tag{62}$$

If we choose $z_0=0$ and simplify we find:

$$|z|^{1-p/q}\,\mathrm{sign}(z) = (1-p/q)(t-t_0) \tag{63}$$

For any $t_0\geq 0$ this allows us to produce solutions of the form

$$z(t) = \begin{cases} ((1-p/q)(t-t_1))^{\frac{1}{1-p/q}} & t > t_1 \geq 0 \\ 0 & 0 \leq t < t_1 \end{cases} \tag{64}$$

Or even more generally:

$$z(t) = \begin{cases} -((1-p/q)(t_0-t))^{\frac{1}{1-p/q}} & t \leq t_0 \leq 0 \\ 0 & t_0 < t < t_1 \\ ((1-p/q)(t-t_1))^{\frac{1}{1-p/q}} & t \geq t_1 \geq 0 \end{cases} \tag{65}$$

For any choices of $t_0\leq 0$ and $t_1\geq 0$ this gives a solution to the ODE which also satisfies the initial condition. Hence there are infinitely many solutions to the initial value problem when $p/q<1$. Note that despite the solution being defined piecewise, it is indeed continuously differentiable when $p/q<1$. For example:

$$z(t_1^+) = (1-p/q)^{\frac{1}{1-p/q}} \lim_{t\to t_1+} (t-t_1)^{\frac{1}{1-p/q}} = 0$$
$$z'(t_1^+) = -(1-p/q)^{\frac{p/q}{1-p/q}} \lim_{t\to t_1+} (t-t_1)^{\frac{p/q}{1-p/q}} = 0 \quad (66$$

For example, with p/q=1/2, t_0=-3 and t_1=2 we have the following solution:

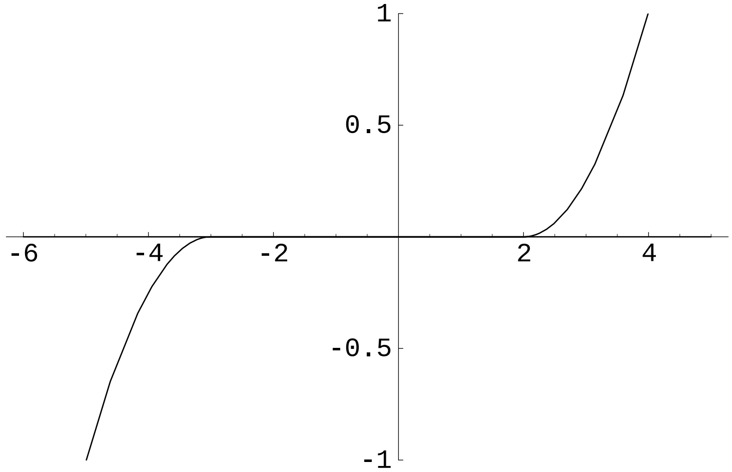

Figure 2

c)

$$\frac{dz}{dt} = |z|^{p/q}$$
$$p > q \quad (67)$$

Solutions of the form demonstrated in part (b) are no longer allowable since for p/q>1 we have:

$$|z(t_1^+)| = |1-p/q|^{\frac{1}{1-p/q}} \lim_{t\to t_1+} |t-t_1|^{\frac{-1}{p/q-1}} = \infty$$
$$|z'(t_1^+)| = |1-p/q|^{\frac{p/q}{1-p/q}} \lim_{t\to t_1+} |t-t_1|^{\frac{-p/q}{p/q-1}} = \infty \quad (68)$$

So, while the solutions are valid for t> t_0 and for t< t_0, they are not valid for t=$t_0$because in general, solutions blow up in finite time. Hence the only solution to the initial value problem is for t_0=∞, i.e. z(t)=0 for all t. This can also be seen by examining the flows of the vector field corresponding to the ODE:

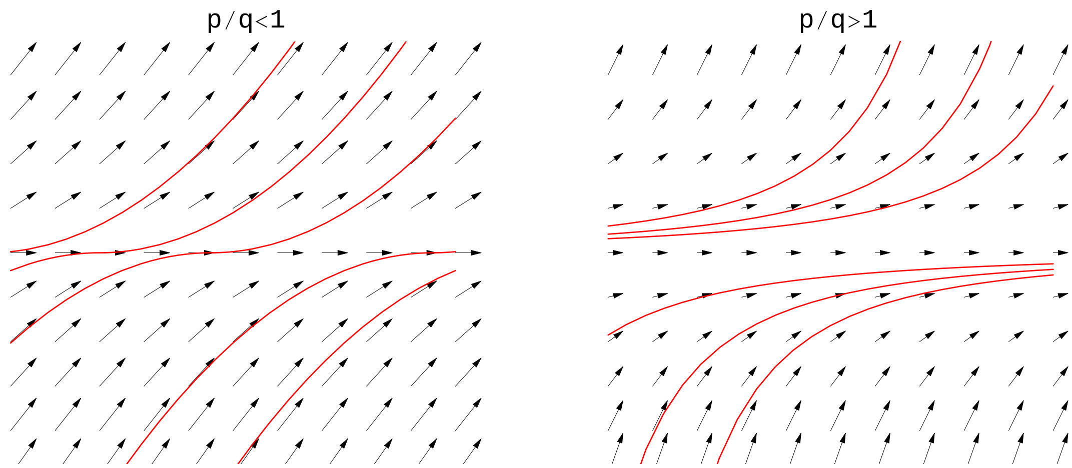

Figure 3

When p/q<1, the trivial flow z=0 can be pieced together with flows with z>0 and z<0 to create an arbitrary number of solutions which all stay finite for finite time. Formulas for these solutions were shown expicitly above. However, for p/q>1 all flows with z>0 or z<0 require infinite time (forward or backward) to reach z=0 and they also become unbounded in finite time. To see this analytically, examine again the general solution formula:

$$\frac{|z|^{1-p/q}}{1-p/q}\,\mathrm{sign}(z) - \frac{|z_0|^{1-p/q}}{1-p/q}\,\mathrm{sign}(z_0) = t - t_0 \tag{69}$$

For p/q>1 if we attempt to set $z_0=0$ we find that the term $|z_0|^{1-p/q}$ becomes infinite, requiring that z=0 for all time or that |t|=∞. Hence, no solution which is initially non-zero can become zero in finite time. The only solution that ever has z=0 is the trivial solution z(t)=0.

hence it is not possible to incorporate these flows with the z=0 flow. This means that z=0 is the only solution.

d)

A function f(·) is said to Lipshitz continuous in (a,b) if the following condition holds for all x,y∈(a,b)

$$|f(x) - f(y)| \le L\,|x - y| \tag{70}$$

The constant L is called the Lipshitz constant and usually depends on the interval (a,b).
As discussed in class, a unique solution to the ODE

$$\frac{dy}{dx} = f(x, y) \tag{71}$$

is guaranteed to exist on any interval where f is Lipshitz continuous in y, i.e. the following condition holds:

$$|f(x, y) - f(x, z)| \le L\,|y - z| \tag{72}$$

For our problem we have:

$$|f(x, z) - f(x, 0)| = |z|^{p/q} \tag{73}$$

If p<q, then the term $|z|^{p/q}$ is larger than |z| for all z∈(-1,1). Indeed,

$$\lim_{z\to 0} \frac{|z|^{p/q}}{|z|} = \infty \tag{74}$$

So there is no Lipshitz constant L that would allow us to make the following type of bound near z=0:

$$|z|^{p/q} < L\,|z| \tag{75}$$

Since this type of bound is impossible, the function f(x,z) isn't Lipshitz continuous near z=0 for p<q. Since the Lipshitz condition doesn't hold near z=0, the uniqueness proof discussed in class doesn't apply. Hence for p<q there is no guarantee that a unique solution exists. This is consistent with what you found in part (b).

If p>q we have the following for all z∈(-1,1):

$$|f(x, z) - f(x, 0)| = |z|^{p/q} \le |z| \tag{76}$$

This means that f(x,z) is Lipshitz continuous at z=0. So a unique solution to

$$\begin{aligned} dz/dt &= |z|^{p/q} \\ z(0) &= 0 \end{aligned} \tag{77}$$

is guaranteed to exist in some interval about t=0. This agrees with the result we found in part (c).

Problem 7 (5x4 points)

The object of this wordy problem is to give you practice in finding exact solutions to one common type of ODE, and also show you that approximate solutions are often much more useful than exact solutions if you want to *understand* what is going on.

If the Mars Exploration Rover folks at JPL had taken only one week of ACM95b, they might unwisely have decided to send the Rovers down radially (i.e. vertically into Mars' atmosphere) because they did not yet know how to solve the systems of coupled nonlinear differential equations that would govern an oblique approach. Consider this simplified radial problem. Let z denote height above Mars' surface, and $\rho = \rho_0 \exp[-z/H]$ the density of its atmosphere, whose surface density is $\rho_0=2\times10^{-5}$ g cm^{-3} and whose scale height is H=11km. Let the Rover (with heat shield!) have constant mass m and effective area A_e (note for cognoscenti, unimportant for this problem: $A_e= A c_D/2$ where A is the actual cross-sectional area, and c_D~2 is the drag coefficient). The drag force on a Rover moving straight down through the atmosphere at speed v=-dz/dt is $F_D=\rho v^2 A_e$. Also acting on the Rover is the force of Mars' gravity (g(0)=373cm s^{-2} at the surface, with vertically integrated escape velocity $v_e(0) = \left[2\int_0^\infty g(z)\,dz\right]^{1/2}$ from Mars of 5km s^{-1}).

a) Show that the Rover's equation of motion is

$$-m\frac{dv}{dt} = -m\,g(z) + v^2\,\rho_0\,A_e\,e^{-z/H} \tag{78}$$

Show also that the left hand side of this equation can be written as mv(dv/dz)=m $d(v^2/2)/dz$, thus changing the independent variable from time t to height z above the surface of Mars. Simplify the equation noting that only the combined quantity $m/(\rho_0 A_e)\equiv l_s$ (the 'stopping length in the Martian atmosphere') appears. Notice that this is a first order linear ODE in v^2.

b) First pretend g(z)=0. Solve the resulting homogeneous equation for $v^2(z)$, given a speed of incidence on the atmosphere $v(\infty)=v_\infty$. Your expression for v(z) should involve only (elementary functions and integral of) H, l_s, v_∞ and z.

c) Now allow a $g(z)\neq0$. Use your result in (b) to find the general solution for $v^2(z)$, given a speed of incidence $v(\infty)=v_\infty$. Your expression for v(z) should involve only (elementary functions and integral of) H, l_s, v_∞ and z.

d) Show that the complicated expression you found in (c) has two simple limiting cases, $H/l_s \ll 1$ and $H/l_s \gg 1$. For each of these two limits, find the lowest order approximation expressions for the Rover landing velocity v(z=0). Also explain how you could have derived these two limiting answers immediately by inspection of the differential equation you found in part (a).

e) To avoid catastrophic destruction of the Rover, you should find from (d) that you want to be in the $H/l_s \gg 1$ limit. What is the maximum radius (in centimeters) a spherical Rover of mean density 1g cm^3 could have if it is to slow down to v(0)<10m s^{-1}? Take v_∞=5km s^{-1}. If the answer makes you worried you now understand why the actual Rover came in obliquely, and had a parachute and retro rockets.

Solution to Problem 7

a) We know from basic physics that:

$$\text{mass}\times\text{acceleration} = \sum \text{Forces} \tag{79}$$

Since the speed of the Rover is given by v=-dz/dt, the acceleration is $d^2z/dt^2=-dv/dt$. Also, the only forces assumed to be acting on the rover are gravity and drag. We then find:

$$-m\frac{dv}{dt} = \text{gravity} + F_D = -m\,g(z) + v^2\,\rho\,A_e = -m\,g(z) + v^2\,\rho_0\,A_e\,e^{-z/H} \tag{80}$$

If the Rover's speed were to ever reach 0, the drag force would vanish and gravity would increase the speed. This intuitive idea shows that the speed of the Rover can never change sign and hence the Rover never reverses directions (i.e. 'bounces' off the atmosphere) hence v is indeed a single valued function of z, so we may change variables from v(t) to v(z). Using the chain rule for derivatives we find:

$$-m\frac{dv}{dt} = -m\frac{dv}{dz}\frac{dz}{dt} = m\,v\frac{dv}{dz} = m\frac{d}{dv}(v^2/2)\frac{dv}{dz} = m\frac{d}{dz}(v^2/2) \tag{81}$$

This gives:

$$m\frac{d}{dz}(v^2/2) = -m\,g(z) + v^2\,\rho_0\,A_e\,e^{-z/H} \tag{82}$$

Divide both sides by m/2:

$$\frac{d v^2}{d z} = -2\, g(z) + \frac{2\, \rho_0\, A_e}{m}\, v^2\, e^{-z/H} = -2\, g(z) + \frac{2}{l_s}\, v^2\, e^{-z/H} \tag{83}$$

Letting $w = v^2$ we have a linear ODE for w(z)

$$\frac{d w}{d z} = \frac{2\, e^{-z/H}}{l_s}\, w - 2\, g(z) \tag{84}$$

b)

$$\frac{d w}{d z} - \frac{2\, e^{-z/H}}{l_s}\, w = 0 \qquad w(\infty) = {v_\infty}^2 \tag{85}$$

In general, an integrating factor for an equation of the following form:

$$y' + f(x)\, y = g(x) \tag{86}$$

Is given by

$$I = e^{\int f(x)\, dx} \tag{87}$$

This allows the ODE to be rewritten in a form easy to integrate:

$$\left(e^{\int f(x)\, dx}\, y\right)' = g(x)\, e^{\int f(x)\, dx} \tag{88}$$

An integrating factor for our problem is:

$$I = e^{\int -\frac{2\, e^{-z/H}}{l_s}\, dz} = e^{\frac{2\, H}{l_s}\, e^{-z/H}} \tag{89}$$

This gives:

$$\left(e^{\frac{2\, H}{l_s}\, e^{-z/H}}\, w\right)' = 0 \tag{90}$$

Solving gives:

$$w = A\, e^{-\frac{2\, H}{l_s}\, e^{-z/H}} \tag{91}$$

The initial condition gives:

$${v_\infty}^2 = w(\infty) = A\, e^0 = A \tag{92}$$

So the solution is:

$$v^2 = {v_\infty}^2\, e^{-\frac{2\, H}{l_s}\, e^{-z/H}} \tag{93}$$

c)

As shown in class the general solution to a linear homogeneous ODE may be used to find the general solution to the corresponding linear inhomogeneous ODE using a technique known as variation of parameters. The basic idea for a first order linear ODE is given below:

We want to solve

$$\frac{d y}{d x} + f(x)\, y = g(x) \tag{94}$$

Suppose that z(x) is any solution to:

$$\frac{dy}{dx} + f(x)\,y = 0 \tag{95}$$

Set y=z(x)w(x) and plug into the inhomogeneous equation:

$$w\,\frac{dz}{dx} + z\,\frac{dw}{dx} + f(x)\,z\,w = g(x) \tag{96}$$

Group terms:

$$w\left(\frac{dz}{dx} + f(x)\,z\right) + z\,\frac{dw}{dx} = g(x) \tag{97}$$

Since z is a solution to the homogeneous ODE, the term in parenthesis is zero. And we are left with a very simple ODE for w:

$$\frac{dw}{dx} = \frac{g(x)}{z(x)} \tag{98}$$

This is solved by simply integrating both sides:

$$w = C + \int \frac{g(x)}{z(x)}\,dx \tag{99}$$

So the general solution to the Inhomogeneous problem is:

$$y = z\,w = z(x)\left(C + \int \frac{g(x)}{z(x)}\,dx\right) \tag{100}$$

In our case, we found that

$$v^2 = v_\infty{}^2\, e^{-\frac{2H}{l_s}\,e^{-z/H}} \tag{101}$$

Solves:

$$\frac{dv^2}{dz} - \frac{2}{l_s}\,e^{-z/H}\,v^2 = 0 \tag{102}$$

We want to use this solution to solve the inhomogeneous problem:

$$\frac{dv^2}{dz} - \frac{2}{l_s}\,e^{-z/H}\,v^2 = -2\,g(z) \tag{103}$$

Applying the formula derived using variation of parameters we have

$$v^2 = v_\infty{}^2\, e^{-\frac{2H}{l_s}\,e^{-z/H}}\left(C + \int_\infty^z \frac{-2\,g(y)}{v_\infty{}^2\,e^{-\frac{2H}{l_s}\,e^{-y/H}}}\,dy\right) \tag{104}$$

The integration constant C is found by applying the initial condition:

$$v_\infty{}^2 = v^2(\infty) = v_\infty{}^2\left(C + \int_\infty^\infty \frac{-2\,g(y)}{v_\infty{}^2\,e^{-\frac{2H}{l_s}\,e^{-y/H}}}\,dy\right) = C\,v_\infty{}^2 \tag{105}$$

Hence the solution is:

$$v^2 = v_\infty{}^2\, e^{-\frac{2H}{l_s}\,e^{-z/H}} + 2\int_z^\infty g(y)\,e^{\frac{2H}{l_s}\,(e^{-y/H}-e^{-z/H})}\,dy \tag{106}$$

Which shows that the surface speed obeys:

$$v^2(0) = v_\infty{}^2\, e^{-\frac{2H}{l_s}} + 2\int_0^\infty g(y)\, e^{\frac{2H}{l_s}(e^{-y/H}-1)}\, dy \tag{107}$$

d) Suppose $H/l_s \ll 1$. This is the case when the atmosphere has very little effect on the Rover and the Rover free falls as if it were in a vacuum. Notice that this gives:

$$\begin{aligned} e^{-\frac{2H}{l_s}e^{-y/H}} &\approx e^0 = 1 \\ e^{-\frac{2H}{l_s}} &\approx e^0 = 1 \end{aligned} \tag{108}$$

The solution formula is then greatly simplified:

$$v^2(0) = v_\infty{}^2 + 2\int_0^\infty g(y)\, dy = v_\infty{}^2 + v_e{}^2(0) \tag{109}$$

Suppose instead that $H/l_s \gg 1$. This is the case when drag forces are strong enough to decelerate the Rover very rapidly resulting in a Rover that falls at the local terminal velocity. Notice that this gives:

$$e^{-\frac{2H}{l_s}} \approx e^{-\infty} = 0 \tag{110}$$

The solution is then of the form:

$$v^2(0) \approx 2\int_0^\infty g(y)\, e^{-\frac{2H}{l_s}(1-e^{-y/H})}\, dy \tag{111}$$

The integrand will be nearly zero everywhere away from y=0. Hence this integral may be replaced with

$$g(0)\int_0^\infty e^{-\frac{2}{l_s}y}\, dy \tag{112}$$

and the error of this approximation is of lower order (see Bender & Orszag for more on the asymptotic approximation of integrals). So our approximation is:

$$v^2(0) \approx 2\, g(0)\int_0^\infty e^{-\frac{2}{l_s}y}\, dy = l_s\, g(0) \tag{113}$$

In part (a) we derived the equation:

$$\frac{dv^2}{dz} - \frac{2}{l_s}\, e^{-z/H}\, v^2 = -2\, g(z) \tag{114}$$

If we rescale z by setting z=Hy we have:

$$\frac{dv^2}{dy} - \frac{2H}{l_s}\, e^{-y}\, v^2 = -2\, H\, g(y\, H) \tag{115}$$

Notice that the term H/l_s appears explicitly in this equation. If $H/l_s \ll 1$, we should eliminate the drag term so that the rover falls as if in a vacuum. This is done by setting $H/l_s=0$:

$$\frac{dv^2}{dy} = -2\, H\, g(y\, H) \tag{116}$$

Integrating and applying the initial condition gives:

$$v^2 = v_\infty{}^2 + 2\int_{z/H}^\infty g(z)\, dz \tag{117}$$

Setting z=0 we find the same answer as above:

$$v^2(0) = v_\infty{}^2 + v_e{}^2$$

If instead $H/l_s \gg 1$ we might write:

$$\frac{l_s}{H}\frac{dv^2}{dy} - 2e^{-y}v^2 = -2 l_s g(yH) \quad (119)$$

In this limit we should eliminate the acceleration term so that the rover falls at the terminal velocity. This is done by setting l_s/H=0 and solving for v:

$$v^2(0) = l_s g(0) \quad (120)$$

This also agrees with the approximation found above.

e) In the previous part we used two different approaches to show

$$v^2(0) \approx \begin{array}{ll} v_\infty{}^2 + v_e{}^2 & H/l_s \ll 1 \\ l_s g(0) & H/l_s \gg 1 \end{array} \quad (121)$$

From the problem statement we know the following:

$$\begin{aligned} v_\infty{}^2 &= 25\,\text{km}^2\,\text{s}^{-2} = 2.5\times10^{11}\,\text{cm}^2\,\text{s}^{-2} \\ v_e{}^2(0) &= 25\,\text{km}^2\,\text{s}^{-2} = 2.5\times10^{11}\,\text{cm}^2\,\text{s}^{-2} \\ l_s &= \frac{m}{\rho_0 A_e} \\ m &= \text{volume} * \text{density} = \left(\frac{4}{3}\pi r^3\right)\text{g} \\ \rho_0 &= 2\times10^{-5}\,\text{g cm}^{-3} \\ A_e &= A\,c_D/2 \sim A = \pi r^2\,\text{cm}^2 \\ g(0) &= 373\,\text{cm s}^{-2} \end{aligned} \quad (122)$$

These values give:

$$v^2(0) \approx \begin{array}{ll} 5\times10^{11}\,\text{cm}^2\,\text{s}^{-2} & H/l_s \ll 1 \\ (\frac{746}{3}\times10^5\,r)\,\text{cm}^2\,\text{s}^{-2} & H/l_s \gg 1 \end{array} \quad (123)$$

When $H/l_s \ll 1$ the velocity at the surface is over 7000m/s. Much too fast! So the limit we are interested in is H/l_s>>1. If we desire the velocity to be below 10m/s=10^3cm/s we have the following inequality:

$$\sqrt{\frac{746}{3}\times10^5\,r} < 10^3 \quad (124)$$

This gives:

$$r < \frac{30}{746} = 0.040\ldots \quad (125)$$

In order to have the desired velocity, the Rover would need to be less than 0.5mm! Obviously a free-falling radial approach isn't the right idea.

For completeness note that

$$H/l_s = \frac{33}{2r} \quad (126)$$

For r=0.040.. this is indeed $\gg$1, so our approximation is valid.

A note from Dr. Phinney to the students:

THE MORAL OF THIS EXERCISE (especially for those of you who had trouble with the approximations to the exact solution: IT IS ALMOST ALWAYS EASIER TO FIND AND UNDERSTAND SOLUTIONS TO APPROXIMATE EQUATIONS THAN IT IS TO FIND AND UNDERSTAND THE CORRESPONDING LIMITS OF 'EXACT" SOLUTIONS TO 'EXACT" GENERAL EQUATIONS. Remembering this in later life may save you a lot of blood, sweat, tears and money.