ACM95b/100b Lecture Notes

Caltech 2004

Laplace Transform

$$\mathcal{L}\{f(t)\} \equiv F(s) \equiv \int_0^\infty e^{-st} f(t)\,dt \tag{1}$$

Examples
a) $f(t) = 1, \;\; t \geq 0$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{b\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^b = \frac{1}{s}$$

b) $f(t) = e^{at}, \;\; t \geq 0$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a}, \;\; s > a$$

Existence
The Laplace transform exists for $s > a$ if:

(1a) f is piecewise continuous on $0 \leq t \leq M$

(2a) f is of exponential order as $t \to \infty$
(i.e. $|f(t)| \leq Ke^{at}, \;\; t \geq M$, with K, a, M constant)

Writing

$$\int_0^\infty e^{-st} f(t)\,dt = \int_0^M e^{-st} f(t)\,dt + \int_M^\infty e^{-st} f(t)\,dt, \tag{2}$$

the first integral exists by (1a) and $|e^{-st}f(t)| \leq Ke^{(a-s)t}$ for $t \geq M$ so if $s > a$, the second integral exists by (2a).

Linearity

$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\} \tag{3}$$

Transforms of Derivatives
For existence of $\mathcal{L}\{f'(t)\}$ for $s > a$ we require

(1b) f is continuous with f' piecewise continuous on $0 \leq t \leq M$

(2b) f is of exponential order as $t \to \infty$.
(i.e. $|f(t)| \leq Ke^{at}, \;\; t \geq M$, with K, a, M constant)

If the discontintuities in $f'(t)$ are located at $t_1, t_2, \cdots t_n$ then

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = \underbrace{\left\{\int_0^{t_1} + \int_{t_1}^{t_2} + \cdots + \int_{t_n}^{\infty}\right\}}_{\text{integrate by parts}} e^{-st} f'(t)\,dt \tag{4}$$

$$= e^{-st} f(t) \underbrace{\left\{\Big|_0^{t_1} + \Big|_{t_1}^{t_2} + \cdots + \Big|_{t_n}^{\infty}\right\}}_{\text{telescope using (1b)}} + s\int_0^\infty e^{-st} f(t)\,dt \tag{5}$$

$$= \underbrace{e^{-st} f(t)\Big|_0^\infty}_{\text{use (2b) and } s>a} + s\underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{\mathcal{L}\{f(t)\}} \tag{6}$$

$$= -f(0) + s\mathcal{L}\{f(t)\}, \qquad s > a \tag{7}$$

$$\boxed{\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0), \qquad s > a}$$

For existence of $\mathcal{L}\{f^{(n)}(t)\}$ for $s > a$ we require

(1c) $f(t), \cdots, f^{(n-1)}(t)$ is continuous with $f^{(n)}$ piecewise continuous on $0 \le t \le M$

(2c) $f(t), \cdots, f^{(n-1)}(t)$ are of exponential order as $t \to \infty$.
(i.e. $|f(t)|, \cdots, |f^{(n-1)}(t)| \le Ke^{at},\ t \ge M$, with K, a, M constant)

and by induction we obtain

$$\boxed{\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0), \qquad s > a}$$

Note that if $f(t), \cdots, f^{(n-1)}(t)$ vanish at $t = 0$ then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} \tag{8}$$

so differentiation in t space corresponds to multiplication in s space.

Convolution Integrals

Suppose $f(t)$, $g(t)$, and $h(t)$ have transforms $F(s)$, $G(s)$, and $H(s)$ for $s > a \ge 0$. If

$$H(s) = F(s)G(s) \tag{9}$$

then $h(t)$ is the *convolution* of $f(t)$ and $g(t)$

$$\begin{aligned} h(t) &= f * g = \textstyle\int_0^t f(t-\tau)g(\tau)d\tau, \\ &= g * f = \textstyle\int_0^t f(\tau)g(t-\tau)d\tau. \end{aligned} \tag{10}$$

Hence, if $H(s)$ can be expressed as a product of known transforms, $h(t)$ can be expressed as a convolution integral. To sketch the proof of (10) we start with the product of transforms (9) and seek to identify the form of $h(t)$.

$$H(s) = F(s)G(s) = \int_0^\infty e^{-s\xi} f(\xi)\, d\xi \;\cdot\; \int_0^\infty e^{-s\tau} g(\tau) d\tau \tag{11}$$

$$= \int_0^\infty g(\tau) \left[\int_0^\infty e^{-s(\xi+\tau)} f(\xi)\, d\xi \right] d\tau \tag{12}$$

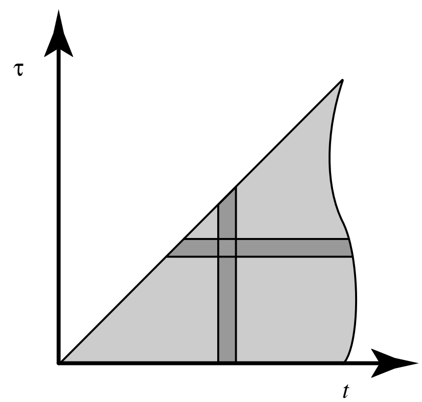
(let $\xi{=}t{-}\tau$ so $d\xi{=}dt$)

$$= \int_0^\infty g(\tau) \left[\int_\tau^\infty e^{-st} f(t-\tau)\, dt \right] d\tau \tag{13}$$

(reverse order of integration over triangular region)

$$= \int_0^\infty e^{-st} \underbrace{\left[\int_0^t f(t-\tau) g(\tau) d\tau \right]}_{\mathcal{L}^{-1}\{H(s)\} \text{ by definition}} dt \tag{14}$$

$$= \int_0^\infty e^{-st} h(t)\, dt \tag{15}$$



Reverse order of integration over triangular region