

ACM95b/100b Lecture Notes

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Introduction to Differential Equations

Differential Equations (DEs) specify (families of) functions by the relations between their derivatives. Arise in nature as describing where things will be/how much of them there will be by simple functions of how they just were, and how they were changing. Most common are 1st and 2nd derivatives, but sometimes (elasticity, motion of molasses, continents) higher, e.g. 4th derivs. A couple of common examples:

- a) Dynamics; Newton's second law

$$ma = m \frac{d^2x}{dt^2} = F(x)$$

Ballistics

$$ma = m \frac{d^2x}{dt^2} = F(x, dx/dt)$$

- b) Chemical or nuclear reaction networks, population dynamics

$$\frac{dy_i}{dt} = \sum_{j,m} k_{jm,i} y_j y_m - \sum_{j,m} k_{ij,m} y_i y_j - k_i y_i$$

(considering 2-body reactions only, reaction rate coeffs $k_{jm,i}$ for reaction $j + m \rightarrow i + x$, plus spontaneous decay/death rate k_i). Examples: ozone+CFCs in stratosphere; isotopes and elements in cores of stars, predator-prey populations.

Definition: Ordinary Differential Equations (ODEs) have only one indep variable (often time in applications), e.g.:

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = \sin t$$

e.g. chemical reactions if well mixed.

Definition: Partial Differential Equations (PDEs) have more than one indep (often space and time in applications), e.g.:

$$\frac{\partial \psi}{\partial t} = \nabla \cdot (D \nabla \psi) = D \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) + S(x, y, z)$$

(if D constant) e.g. chemical reactions if not well mixed, also have diffusion of species due to concentration gradients.

Definition of Solution of DE

$$f(t, y, y', \dots, y^{(n)}) = 0$$

n th order DE. A solution of this eqn is defined to be a real (or complex if desired) valued function $y(t)$ defined over some interval I (open, closed, half-open or infinite) with the two properties

1. $y(t)$ and its first n derivatives exist for each t in I , hence y and its first $n - 1$ derivatives are continuous in I .
2. Substitution of $y(t)$ into DE makes the equation an identity in t in interval I —i.e. the equation holds for each t in I .

Solutions and Constraints Any single solution of the DE is called a particular solution. The set of all solutions is called the “general solution”. In the y, t plane, a particular solution is a curve, called an integral curve. The general solution is the set (family) of all integral curves.

For ODE’s two types of constraints for picking out particular solutions are most common:

- a) **Initial value problem (IVP)**. Initial conditions specified at a single point.

$$y'' + y = 0$$

subject to $y(0) = 1, y'(0) = 3$. For continuous functions (away from singularities), there are existence/ uniqueness theorems for IVPs.

- b) **Boundary value problem (BVP)**. Specify boundary conditions at two or more points.

$$y'' + y = 0$$

subject to $y(0) = 1, y'(2\pi) = -1$ BVPs may not have solutions, or if they do, solutions may be nonunique. *Not* a mathematical curiosity: of considerable physical importance (cf eigenvalue problems).

Classification of ODEs

Order of DE is order of highest derivative appearing.

$$F(t, y, y', \dots, y^{(n)}) = 0$$

is n th order.

ODE is **linear** if F is linear function of dependent variable y and its derivatives. Otherwise it is **nonlinear**.

Linear ODE of order N has form (Euler-Lagrange theorem!)

$$\sum_{n=0}^N a_n(t) \frac{d^n y}{dx^n} = G(t)$$

If $G(t) = 0$ for all values of t relevant to solution of interest, equation called **homogeneous**.

If $G(t) \neq 0$ for any value of t relevant to solution of interest, equation called **nonhomogeneous**.

Related subjects: difference equations (what computers actually solve when you ask them to solve differential equations numerically), integral equations.

Special functions: $\ddot{y} + y = 0$ arises a lot, and its solutions are ‘elementary functions’ defined by power series. Lots of other ODEs arise so often in physics that their solutions are as well studied and efficiently computed that they get names (but not so far calculator buttons) —e.g. Bessel functions, Airy functions, elliptic functions, etc. As with trig functions, there are other ways to define them than by ODEs.