ACM 95b/100b Handout 3/3/2004: Greens Functions

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(1)

DE $Ly = f(x)$ $\quad Ly = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - \overbrace{q(x)y + \lambda w(x) y}^{q(x)y}$ the S-L rule

BC $B_a y = 0 \qquad B_a y \equiv \alpha_1 y(a) + \beta_1 y'(a)$

$B_b y = 0 \qquad B_b y \equiv \alpha_2 y(b) + \beta_2 y'(b)$

boundary operators

α_1, β_1 given consts, not both 0

α_2, β_2 " " " " "

to be solved on $a \le x \le b$ with p, q continuous, $p \neq 0$ in $[a, b]$

Notice boundary operators are linear

<u>Green's function</u> $G(x; \xi)$ is solution to

(2)

$$\boxed{\begin{aligned} &LG = \delta(x-\xi) \\ &B_a G = 0 \\ &B_b G = 0 \end{aligned}}$$

Notice that it depends only on the homogeneous solutions y_1 and y_2 and the parameter ξ.

Can linearly superpose $G(x, \xi)$ for different ξ to get solution to (1) for any $f(x)$:

(3)

$$\boxed{\underset{\text{solves (1)}}{y(x)} = \int_a^b \underset{\text{RHS of (1)}}{f(\xi)}\, \underbrace{G(x;\xi)}_{\text{solves (2)}}\, d\xi}$$

<u>Check:</u>

$\boxed{\text{DE}}$

$$Ly = \int_a^b f(\xi)\, \underbrace{L\,G(x,\xi)}_{\text{by (2)}}\, d\xi$$

$$= \int_a^b f(\xi)\, \delta(x-\xi)\, d\xi$$

$$= f(x) \checkmark$$ so solves DE (1)

$\boxed{\text{BC}}$

$$B_a y = \int_a^b f(\xi)\, \underbrace{B_a G}_{0 \text{ by (2)}}\, d\xi = 0 \checkmark$$

$$B_b y = \int_a^b f(\xi)\, \underbrace{B_b G}_{0 \text{ by (2)}}\, d\xi = 0 \checkmark$$

Ways to find $G(x,\xi)$

** 1. (Best for 2^{nd} order LODEs) : express solution to (2) as linear combinations of solutions to homogeneous equation, matching at ξ $(a<\xi<b)$ to produce the δ-discontinuity from the derivative of $G(x,\xi)$.

2. Eigenfunction expansion (necessary for many PDEs, but not for ODEs where closed form solution of type (1) available)

3. Just solve (1) by variation of parameters compare to (3) and identify G.
[usually more work than method 1!]

<u>Details of method 1.</u> for 2^{nd} order LODEs of form (1):

<u>Two cases</u>:

Case $\boxed{\text{I. } \lambda \text{ is not an eigenvalue}}$ Then (1) has unique solution, and the homogeneous equation $Ly=0$ $B_a y=0$ $B_b y=0$ has only the trivial solution $y=0$.

In this case

(4)
$$\boxed{\begin{array}{ll} \text{Let } y_1(x) \text{ be a solution to } & Ly_1=0 \quad B_a y_1=0 \\ \text{Let } y_2(x) \text{ be a solution to } & Ly_2=0 \quad B_b y_2=0 \end{array}}$$

y_1 and y_2 are linearly indep and unique up to multiplicative constants

(5) Clearly $G(x,\xi)=\begin{cases} c_1(\xi)\, y_1(x) & x<\xi \\ c_2(\xi)\, y_2(x) & x>\xi \end{cases}$

we choose $c_1(\xi)$ and $c_2(\xi)$ to satisfy (2):

(6) $\boxed{G(x,\xi) \text{ must be continuous at } \xi}$, otherwise $\frac{dG}{dx} = \delta$ $\frac{d^2G}{dx^2} = \delta'(x)$

and the RHS of (2) has no δ' $\int_{\xi^-}^{\xi^+} \delta(x-\xi)dx$

but

(7) $\boxed{\frac{dG(x,\xi)}{dx} \text{ must be discontinuous, with } p(x)\frac{dG}{dx}\Big|_{x=\xi^+} - p(x)\frac{dG}{dx}\Big|_{x=\xi^-} = 1}$

so that $\int_{\xi^-}^{\xi^+} \frac{d}{dx}\left(p\frac{dG}{dx}\right)dx = \int_{\xi^-}^{\xi^+} \delta(x-\xi)\,dx$

Solving the resulting pair of equations for $c_1(\xi)$ and $c_2(\xi)$ gives

(8)
$$\boxed{G(x,\xi) = \begin{cases} \dfrac{y_2(\xi)\, y_1(x)}{p(\xi)\, W(\xi)} & x < \xi \\[2ex] \dfrac{y_1(\xi)\, y_2(x)}{p(\xi)\, W(\xi)} & x > \xi \end{cases}}$$

NB this is true for any 2nd order LODE, not just the S-L form (1)

For the S-L form (1) this simplifies;

(9) Abel's Thm $\exists$ $w(x) = e^{-\int \frac{p'}{p} dx} = \frac{c}{p}$, so $\boxed{p(\xi) W(\xi) = \text{const, indep of } \xi \text{ for S-L ODE's}}$

The constant can be evaluated e.g. by setting $\xi = a$ or $\xi = b$

$\boxed{\text{Case II. } \lambda \text{ is an eigenvalue}}$ Then the homogeneous equation (1) has a nontrivial solution - the eigenfunction $y_1(x)$, which satisfies

$Ly = 0$ $\quad B_a y_1 = 0$ $\quad B_b y_1 = 0$

The derivation of G for case I fails now:
if choose y_2 to be solution of $Ly_2 = 0$ $B_b y_2 = 0$,
$y_2 = \text{const} \times y_1$
and the Wronskian $W(y_2, y_1) = 0$ so (8) blows up.

(10) we can nevertheless follow the derivation of (8) using y_2 to be the second (linearly independent of y_1) solution to $Ly_2 = 0$

But now $B_b y_2 \neq 0$ since it is linearly indep of y_1 for which $B_b y_1 = 0$.

(11) we get the same result as (8). Substituting in (3) and evaluating

$y(b)$ and $y'(b)$ we find

suppose b is 0 $\frac{1}{y}(\,)$

$$\rightarrow B_b(y) = \underbrace{B_b(y_2)}_{\neq 0} \int_a^b \frac{y_1(\xi) f(\xi) d\xi}{p(\xi) W(\xi)}$$

$\therefore$ for S-L problem $[pW = \text{const}]$

(12) If λ is an eigenvalue and $y_1(x)$ the associated eigenfunction a solution to (1) exists if and only if $f(x)$ satisfies

$$\boxed{\int_a^b f(\xi)\, y_1(\xi)\, d\xi = 0}$$

i.e. there is no solution unless $f(x)$ is orthogonal to the eigenfunction.
If $f(x)$ does satisfy this orthogonality constraint, then the solution to (1) is <u>not unique</u> (why!) and given by

$$y = c_1 y_1(x) + \int_a^b G(x;\xi) f(\xi) d\xi$$

G given by (8) by with y_2 given by (10), not (4)