Maple worksheet by E.S. Phinney 2/18/2004 ACM95b/100b handout.

Solution to the heat/diffusion equation on the interval $0 \le x \le 1$:

$$\phi_t = \phi_{xx}, \quad \phi(x,0) = f(x), \quad \phi(0,t) = 0, \quad \phi(1,t) = 0 \; . \tag{1}$$

It was shown in class that at given t, $\phi(t,x)$ can be expanded in eigenfunctions $y_n(x)$ of the Sturm-Liouville operator $Ly + \lambda y = 0$, with $y(0) = y(1) = 0$ and $L = d^2/dx^2$. Thus we found

$$\phi(x,t) = \sum_{n=1}^{\infty} a_n(t)\sqrt{2}\sin n\pi x \tag{2}$$

By multiplying the PDE by $y_m(x)$ and integrating by parts over x, we found $da_m/dt = -m^2\pi^2 a_m(t)$, so

$$a_n(t) = a_n(0)\exp(-n^2\pi^2 t) \tag{3}$$

The initial conditions give

$$a_n(0) = 2\int_0^1 f(x)\sin m\pi x \; . \tag{4}$$

So finally

$$\phi(x,t) = \sum_{n=1}^{\infty}\left[2\int_0^1 f(\xi)\sin n\pi\xi d\xi\right]\exp(-n^2\pi^2 t)\sin n\pi x \; . \tag{5}$$

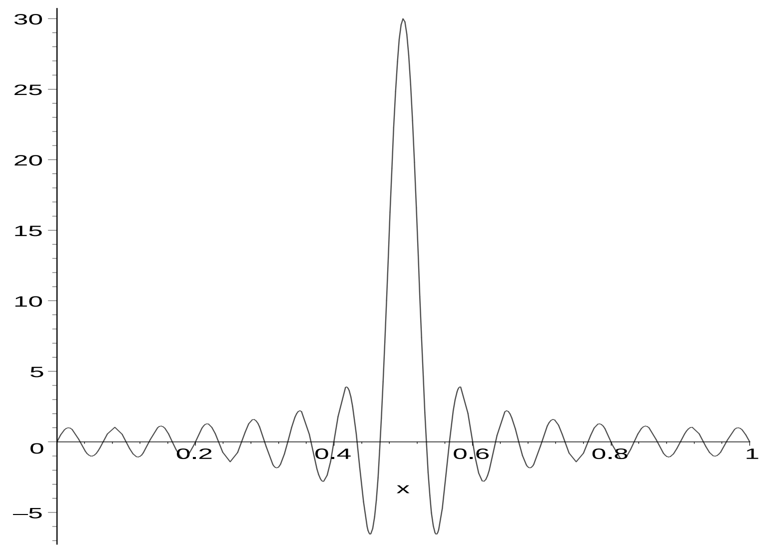
This solution converges to its first Fourier mode remarkably fast. To show this, we set $f(x) = \delta(x - 1/2)$ in our solution -the spikiest possible initial temperature profile (and also the Greens' function for $\xi = 1/2$).

```
> Fourser := (x,t,n) ->
> Sum(2*sin(k*Pi/2)*sin(k*Pi*x)*exp(-k^2*Pi^2*t),k=1..n);
```

$$\mathit{Fourser} := (x,\, t,\, n) \to \sum_{k=1}^{n} (2\sin(\frac{1}{2}\,k\,\pi)\sin(k\,\pi\,x)\,e^{(-k^2\,\pi^2\,t)})$$

If we plot the first 30 (really 15, since all the even k coefficients are zero by symmetry) terms for t=0, we see that this does look as if it is reproducing the delta function.

```
> plot([Fourser(x,0,30)],x=0..1);
```



Adding more terms makes the central spike narrower and higher. One can also verify the fundamental delta property: that it integrates to one: the integral of the series over x for $t = 0$ is $4/\pi(1 - 1/3 + 1/5 - 1/7..) = 1$. The error in the integral if we truncate the sum after n terms is of order $1/n$:

```
> int(Fourser(x,0,30),x=0..1);
```

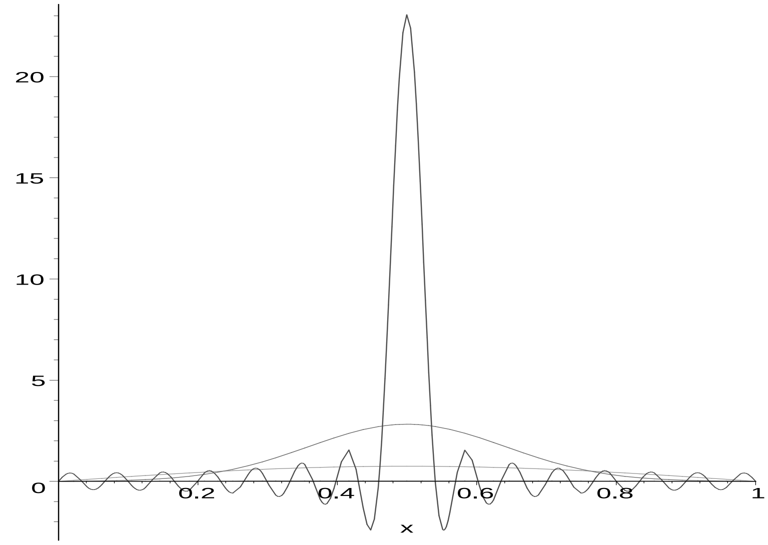
$$2 \frac{\sum_{k=1}^{30} \left(- \frac{\sin(\frac{1}{2} k \pi) (-1 + \cos(k \pi))}{k \sqrt{\pi}} \right)}{\sqrt{\pi}}$$

```
> evalf(%);
```

$$1.021197210$$

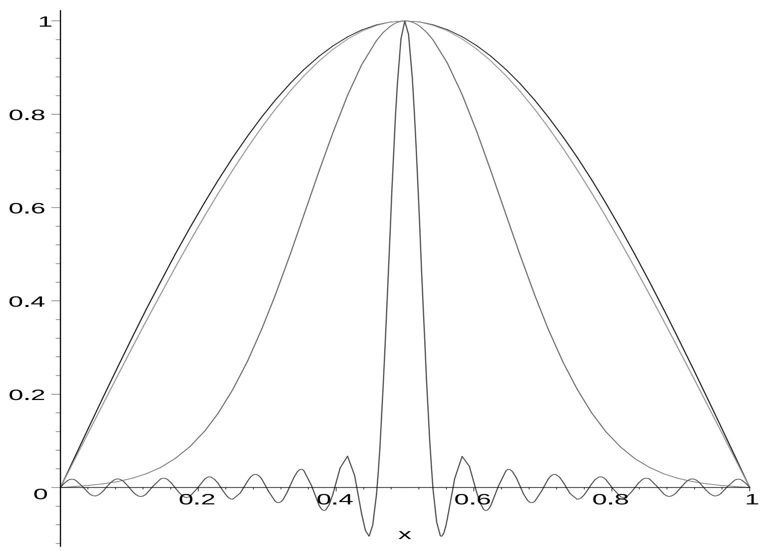
Now let's see what the solution looks like at later times. Notice that the first two terms are $\phi(x,t) = 2\sin(\pi x) exp(-\pi^2 t) - 2\sin(3\pi x) exp(-9\pi^2 t) + ...$ So the second term is becoming negligible compared to the first at times $t > 1/(8\pi^2) = 0.013$. The contribution of the higher terms drops even faster, so the height of the delta (a sum of n 1's $= n$ for $t = 0$) quickly becomes just the first term.

```
> plot([Fourser(x,0.0001,30),
> Fourser(x,0.01,30),Fourser(x,0.1,30),Fourser(x,0.5,30)],x=0..1,color=[
> red,sienna,green,blue]);
```



We can see that the shape of the temperature profile has converged very accurately to $\sin(\pi x)$ (the first term) for $t > 0.05$ by normalising the temperature profiles (which drop rapidly to invisibility in the previous unnormalized graph):

```
> plot([Fourser(x,0.0001,30)/Fourser(0.5,0.0001,30),
> Fourser(x,0.01,30)/Fourser(0.5,0.01,30),Fourser(x,0.05,30)/Fourser(0.5
> ,0.05,30),Fourser(x,0.3,30)/Fourser(0.5,0.3,30)],x=0..1,color=[red,sie
> nna,green,blue]);
```



What were the actual peaks of the t=0 series, and the four times of the previous two plots?

```
> evalf([Fourser(0.5,0,30),Fourser(0.5,0.0001,30),Fourser(0.5,0.01,30),
> Fourser(0.5,0.05,30),Fourser(0.5,0.3,30)]);
```

$$[30., 23.06316230, 2.820947918, 1.244565533, .1035465365]$$