ACM95b Final Exam 2004

Problem 1 (4×6 points)

(a) 6 points

(1) 1 point

Regular S-L

(2) 1 point

Not a S-L problem since y(0)=1 makes the operator non-selfadjoint

(3) 1 point

Singular S-L

(4) 1 point

Periodic S-L

(5) 1 point

Regular S-L

(6) 1 point

Singular S-L

(b) 6 points

$$0 = \frac{1}{r^2} (r^2 u_r)_r + \frac{1}{r^2 \operatorname{Sin}\theta} (\operatorname{Sin}\theta \, u_\theta)_\theta + \frac{1}{r^2 \operatorname{Sin}^2 \theta} u_{\phi\phi} \qquad (1)$$

Set u=R(r)Ξ(θ)Φ(ϕ)

$$\frac{1}{\operatorname{Sin}\theta} \frac{(\operatorname{Sin}\theta \, \Xi')'}{\Xi} + \frac{1}{\operatorname{Sin}^2 \theta} \frac{\Phi''}{\Phi} = -\frac{(r^2 R')'}{R} \qquad (2)$$

So

$$(r^2 R')' + \lambda R = 0 \qquad (3)$$

This equation is of form (5).

(c) 6 points

$$(1 - x^2) y'' - x y' + \lambda y = 0 \qquad (4)$$

Rewrite this as

$$y'' - \frac{x}{1 - x^2} y' + \frac{\lambda}{1 - x^2} y = 0 \qquad (5)$$

After finding the right integration factor this equation will be of the form

$$(p\,y')' + (q + \lambda\,r)\,y = 0 \tag{6}$$

Expanding this

$$y'' + \frac{p'}{p}\,y' + \left(\frac{q + \lambda r}{p}\right) y = 0 \tag{7}$$

So

$$\frac{p'}{p} = -\frac{x}{1 - x^2} \tag{8}$$

or

$$p = \sqrt{1 - x^2} \tag{9}$$

This gives

$$\left(\sqrt{1 - x^2}\;y'\right)' + \frac{\lambda}{\sqrt{1 - x^2}}\,y = 0 \tag{10}$$

(d) 6 points

(i) 2 points

The weight factor is

$$w(x) = \frac{1}{\sqrt{1 - x^2}} \tag{11}$$

and we are considering the interval $x \in [-1,1]$, so the orthogonality condition is

$$\int_{-1}^{1} T_n(x)\,T_m(x)\,\frac{1}{\sqrt{1 - x^2}}\,dx = \begin{matrix} 0 & n \neq m \\ N_n & n = m \end{matrix} \tag{12}$$

Where N_n is some non-zero number that in principle could be calculated.

(ii) 2 points

By the class notes of 2/23/04, this singular S-L problem is not guaranteed to have a complete set of eigenfunctions unless the eigenvalues are discrete (i.e. there is no continuous or residual spectrum). Since this is a singular S-L problem, it is possible that the eigenfunctions are complete but we don't know for certain.

(iii) 2 points

No, completeness isn't guaranteed as explained in part (ii)

Problem 2 (6+4 points)

(a) 6 points

$$\begin{aligned} &(x^2\,y')' + \lambda\,y = 0 \\ &y(1) = y(e) = 0 \end{aligned} \tag{13}$$

Following the hint, set y=x^ν

$$\nu(\nu+1)+\lambda=0 \tag{14}$$

Following the hint set 4λ-1=μ^2. This gives

$$\nu_\pm = \frac{-1 \pm i\,\mu}{2} \tag{15}$$

We check the Wronskian of these two solutions

$$\begin{vmatrix} x^{\nu_+} & x^{\nu_-} \\ \nu_+\, x^{\nu_+} & \nu_-\, x^{\nu_-} \end{vmatrix} = (\nu_- - \nu_+)\, x^{\nu_- + \nu_+} \tag{16}$$

This is non-zero on [1,e] provided that $\mu \neq 0$. In this case, the general solution is of the form

$$y = A\, x^{\nu_+} + B\, x^{\nu_-} \tag{17}$$

If μ=0 (i.e. λ=1/4) then the we only have on linearly independent solution

$$x^{-1/2} \tag{18}$$

To find another we use reduction of order. Set

$$y(x) = x^{-1/2}\, z(x) \tag{19}$$

Pluging into the ODE with λ=1/4 gives:

$$(x\, z')' = 0 \tag{20}$$

So (3 points)

$$y = x^{-1/2}\,(A + B \ln x) \tag{21}$$

Now let us try to fit the boundary conditions.

Case 1: $\lambda \neq 1/4$ (2 points)

$$\begin{aligned} 0 &= y(1) = A + B \\ 0 &= y(e) = A\, e^{\nu_+} + B\, e^{\nu_-} \end{aligned} \tag{22}$$

A homogeneous linear system has a non-trivial solutions iff its determinant is zero. So we require

$$0 = \begin{vmatrix} 1 & 1 \\ e^{\nu_+} & e^{\nu_-} \end{vmatrix} = e^{\nu_-} - e^{\nu_+} \tag{23}$$

This is satisfied if

$$\nu_+ = \nu_- + 2\pi n\, i \tag{24}$$

for integer n. In terms of μ this requirement is

$$\mu = 2\pi n \tag{25}$$

The eigenfunctions are of the form

$$y = A\,(x^{\nu_+} - x^{\nu_-}) = A\, x^{-1/2}\,(x^{\pi n i} - x^{-\pi n i}) \tag{26}$$

Which, using the hint, can be written

$$y_n = A\, x^{-1/2}\,(e^{i\pi n \ln x} - e^{-i\pi n \ln x}) = \alpha\, x^{-1/2}\, \mathrm{Sin}(\pi n \ln x) \tag{27}$$

Case 2: λ=1/4 (1 point)

$$0 = y(1) = A$$
$$0 = y(e) = e^{-1/2}\,(A + B) \tag{28}$$

This gives A=B=0. So $\lambda=1/4$ is not an eigenvalue

We conclude that the eigenvalues and eigenfunctions are

$$\lambda_n = \frac{1}{4}\,(1 + 4\pi^2 n^2)$$
$$y_n = x^{-1/2}\,\mathrm{Sin}(\pi\, n \ln x) \tag{29}$$
$$n = 1, 2, 3, \ldots$$

■ (b) 4 points

■ (i) 1 point

The eigenvalues are

$$\lambda_n = \frac{1}{4}\,(1 + 4\pi^2 n^2) \tag{30}$$

for all positive integer n. Hence there are infinitely many of them. The spacing between eigenvalues is given by

$$\lambda_n - \lambda_m = \pi^2\,(n^2 - m^2) \tag{31}$$

This spacing doesn't approach 0, so the eigenvalues have no accumulation point.

■ (ii) 1 point

As we found in part (a) for each n, there is exactly one eigenfunction up to a multiplicative constant

$$y_n = A_n\, x^{-1/2}\,\mathrm{Sin}(\pi\, n \ln x) \tag{32}$$

■ (iii) 1 point

$$(y_n, y_m) = \int_1^e y_n\, y_m\, dx = \int_1^e x^{-1}\,\mathrm{Sin}(\pi\, n \ln x)\,\mathrm{Sin}(\pi\, m \ln x)\, dx \tag{33}$$

Make the change of integration variable

$$z = \ln x \tag{34}$$

We find

$$(y_n, y_m) = \int_0^1 \mathrm{Sin}(\pi\, n\, z)\,\mathrm{Sin}(\pi\, m\, z)\, dz \tag{35}$$

By what we know from class and homework, these sine functions are orthogonal unless n=m

$$(y_n, y_m) = \begin{matrix} 0 & n \neq m \\ 1/2 & n = m \end{matrix} \tag{36}$$

■ (iv) 1 point

The eigenfunctions will be complete in the space of square integrable functions if for any such function f(x) we have

$$f(x) = \sum_{n=1}^{\infty} A_n\, x^{-1/2}\,\mathrm{Sin}(\pi\, n \ln x) \tag{37}$$

Let's change variables to z=ln x

$$h(z) = e^{z/2}\, f(e^z) = \sum_{n=1}^{\infty} A_n \,\mathrm{Sin}(\pi\, n\, z) \qquad (38$$

We know that the sine function on the right side are complete in the space of square integrable functions on [0,1], so all that's left is to verify that h(z) is square integrable on [0,1]

$$\int_0^1 h^2(z)\, dz = \int_0^1 e^z\, f^2(e^z)\, dz = \int_1^e f^2(x)\, dx \qquad (39)$$

Since we know that f(x) is square integrable on $[1,e]$ this last integral converges and hence h(z) is square integrable on [0,1]. So we conclude that since $\mathrm{Sin}(\pi\, n\, z)$ are complete on [0,1], y_n are complete on $[1,e]$.

Problem 3 (4×6 points)

■ (a) 6 points

$$\begin{aligned} &(x\, G')' = \delta(x - t) \\ &G(0) = \text{finite} \\ &G(1) = 0 \end{aligned} \qquad (40)$$

Method 1: shortcut

It is acceptable for students to quote a formula from a class handout. They should find $y_1 = 1$ and $y_2 = \ln x$.

Method 2: the long way

G will be of the form

$$G = \begin{array}{ll} A + B \ln x & 0 \le x \le t \\ C + D \ln x & t \le x \le 1 \end{array} \qquad (41)$$

The boundary conditions give

$$G = \begin{array}{ll} A & 0 \le x \le t \\ D \ln x & t \le x \le 1 \end{array} \qquad (42)$$

Continuity at x=t requires A= D ln t. The jump condition is found by integrating the ODE from t-ϵ to t+ϵ and letting $\epsilon \to 0$

$$t\, G'(t^+) - t\, G'(t^-) = \lim_{\epsilon \to 0^+} \int_{t-\epsilon}^{t+\epsilon} (x\, G')'\, dx = \lim_{\epsilon \to 0^+} \int_{t-\epsilon}^{t+\epsilon} \delta(x - t)\, dx = 1 \qquad (43)$$

So we require

$$D/t - 0 = 1/t \qquad (44)$$

Using either method, we conclude (5 points)

$$G = \begin{array}{ll} \ln t & 0 \le x \le t \\ \ln x & t \le x \le 1 \end{array} \qquad (45)$$

We suspect that

$$y(x) = \int_0^1 G(x, t)\, f(t)\, dt \qquad (46)$$

will solve

$$\begin{aligned} &(xy')' = f(x) \\ &y(0) = \text{finite} \\ &y(1) = 0 \end{aligned} \qquad (47$$

We verify as follows (1 point)

Mehtod a:

$$\begin{aligned} &y(0) = \int_0^1 G(0, t)\, f(t)\, dt = \ln t \int_0^1 f(t)\, dt = \text{finite} \\ &y(1) = \int_0^1 G(1, t)\, f(t)\, dt = \int_0^1 0\, f(t)\, dt = 0 \\ &(xy')' = \int_0^1 (x\, G_x(x, t))_x\, f(t)\, dt = \int_0^1 \delta(x - t)\, f(t)\, dt = f(x) \end{aligned} \qquad (48)$$

Method b:

$$\begin{aligned} &y = \ln x \int_0^x f(t)\, dt + \int_x^1 \ln t\, f(t)\, dt \\ &y(0) = \lim_{x\to 0}\left(\ln x \int_0^x f(t)\, dt + \int_x^1 \ln t\, f(t)\, dt \right) = \\ &\quad \lim_{x\to 0}\left(\frac{x\, f(x)}{-(\ln x)^2} \right) + \int_0^1 \ln t\, f(t)\, dt < \infty \text{ only if } f(x) \text{ vanishes rapidly enough near } x = 0 \\ &y(1) = \ln 1 \int_0^1 f(t)\, dt + \int_1^1 \ln t\, f(t)\, dt = 0 \\ &(x\, y')' = x\left(\frac{1}{x} \int_0^x f(t)\, dt + x \ln x\, f(x) - x \ln x\, f(x) \right)' = \left(\int_0^x f(t)\, dt \right)' = f(x) \end{aligned} \qquad (49)$$

■ (b) 6 points

$$f(x) = 1 \qquad (50)$$

$$y(x) = \int_0^1 G(x, t)\, dt = y(x) = \ln x \int_0^x dt + \int_x^1 \ln t\, dt = (x \ln x) + (-1 - x \ln x + x) = x - 1 \qquad (51)$$

■ (c) 6 points

Method 1: shortcut (5 points)

As in 3a, students may quote a formula with y_1=Cos x and y_2=Sin(x-1), W(y_1, y_2)=Cos(1), and p(x)=1

Method 2: the long way (5 points)

$$\begin{aligned} &G'' + G = \delta(x - t) \\ &G'(0) = G(1) = 0 \end{aligned} \qquad (52)$$

G will be of the form

$$G = \begin{matrix} A \sin x + B \cos x & 0 \le x \le t \\ C \sin x + D \cos x & t \le x \le 1 \end{matrix} \qquad (53)$$

The boundary conditions give

$$G = \begin{matrix} B \cos x & 0 \le x \le t \\ E \sin(x - 1) & t \le x \le 1 \end{matrix} \qquad (54)$$

Continuity at x=t requires

$$G = \begin{cases} F\,\text{Sin}(t-1)\,\text{Cos}\,x & 0 \le x \le t \\ F\,\text{Sin}(x-1)\,\text{Cos}\,t & t \le x \le 1 \end{cases} \tag{55}$$

The jump condition requires

$$F\,\text{Cos}(t-1)\,\text{Cos}\,t + F\,\text{Sin}(t-1)\,\text{Sin}\,t = 1 \tag{56}$$

So we have

$$G = \begin{cases} \frac{\text{Sin}(t-1)\,\text{Cos}\,x}{\text{Cos}\,1} & 0 \le x \le t \\ \frac{\text{Sin}(x-1)\,\text{Cos}\,t}{\text{Cos}\,1} & t \le x \le 1 \end{cases} \tag{57}$$

We suspect that the solution to

$$\begin{aligned} &y'' + y = f(x) \\ &y'(0) = y(1) = 0 \end{aligned} \tag{58}$$

will be

$$y(x) = \int_0^1 G(x, t)\, f(t)\, dt \tag{59}$$

Regardless of the method used, we verify as follows (1 point)

$$\begin{aligned} y'(0) &= \int_0^1 G_x(0, t)\, f(t)\, dt = \int_0^1 (G_x(x, t))_{x=0}\, f(t)\, dt = \int_0^1 0\, f(t)\, dt = 0 \\ y(1) &= \int_0^1 G(1, t)\, f(t)\, dt = \int_0^1 0\, f(t)\, dt = 0 \\ y'' + y &= \int_0^1 (G_{xx}(x, t) + G(x, t))\, f(t)\, dt = \int_0^1 \delta(x-t)\, f(t)\, dt = f(x) \end{aligned} \tag{60}$$

(d) 6points

When the right boundary is x=π/2, λ=1 is an eigenvalue. That is

$$\begin{aligned} &Ly = y'' + y = 0 \\ &y'(0) = y(\pi/2) = 0 \end{aligned} \tag{61}$$

has the non-trivial solution

$$y_1 = \text{Cos}\,x \tag{62}$$

So the problem

$$\begin{aligned} &y'' + y = f(x) \\ &y'(0) = y(\pi/2) = 0 \end{aligned} \tag{63}$$

doesn't have a solution for arbitrary f(x). It will have a solution only if f(x) is orthogonal to the eigenfunction. We see this as follows. This is regular S-L problem and L is self adjoint. So if y solves the inhomogeneous problem and y_1 is the eigenfunction, then we have

$$(y_1, f) = (y_1, Ly) = (Ly_1, y) = (0, y) = 0 \tag{64}$$

So a solution to the inhomogeneous problem exists only if f is orthogonal to the eigenfunction y_1 i.e.

$$\int_0^1 \text{Cos}(x)\, f(x)\, dx = 0 \tag{65}$$

Problem 4 (4×6 points)

$$u_t = \kappa\, u_{xx} - \sigma u + f(x)$$
$$u(x, 0) = 0 \tag{66}$$

■ (a) 6 points

$$u(x, t) = \sum_{n=-\infty}^{\infty} A_n(t)\, e^{i n x}$$
$$f(x) = \sum_{n=-\infty}^{\infty} f_n\, e^{i n x} \tag{67}$$

By orthogonality we have

$$A_n(t) = \frac{1}{2\pi} \int_0^{2\pi} u(x, t)\, e^{-i n x}\, dx$$
$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, e^{-i n x}\, dx \tag{68}$$

Method 1: The "right" way

Multiply the PDE by the exponential and integrate on [0,2π]

$$\int_0^{2\pi} u_t\, e^{-i n x}\, dx = \kappa \int_0^{2\pi} u_{xx}\, e^{-i n x}\, dx - \sigma \int_0^{2\pi} u\, e^{-i n x}\, dx + \int_0^{2\pi} f e^{-i n x}\, dx \tag{69}$$

Most of these integrals are trivial to evaluate just using the definition of the Fourier coefficients

$$A_n'(t) = \kappa \frac{1}{2\pi} \int_0^{2\pi} u_{xx}\, e^{-i n x}\, dx - \sigma A_n(t) + f_n \tag{70}$$

This last integral can be found by applying integration by parts twice

$$\int_0^{2\pi} u_{xx}\, e^{-i n x}\, dx =$$
$$\left(e^{-i n x}\, u_x + i\, n\, e^{-i n x}\, u\right)_{x=2\pi} - \left(e^{-i n x}\, u_x + i\, n\, e^{-i n x}\, u\right)_{x=0} - n^2 \int_0^{2\pi} e^{-i n x}\, u\, dx = -n^2\, 2\pi A_n(t) \tag{71}$$

So the ODE for A is

$$A_n'(t) = -(n^2 \kappa + \sigma) A_n(t) + f_n \tag{72}$$

Method 2: The "wrong" way (at most one point should be taken off for solving the problem this way)

Plug the expressions for u and f into the PDE

$$\sum_{n=-\infty}^{\infty} A_n'(t)\, e^{i n x} = -\kappa \sum_{n=-\infty}^{\infty} n^2 A_n(t)\, e^{i n x} - \sigma \sum_{n=-\infty}^{\infty} A_n(t)\, e^{i n x} + \sum_{n=-\infty}^{\infty} f_n\, e^{i n x} \tag{73}$$

By orthogonality we have

$$A_n'(t) = -(n^2 \kappa + \sigma) A_n(t) + f_n \tag{74}$$

Regardless of which method we use, the initial condition is found by setting t=0 in the expression for u

$$0 = u(x, 0) = \sum_{n=-\infty}^{\infty} A_n(0)\, e^{i n x} \tag{75}$$

And then using orthogonality to conclude

$$A_n(0) = 0 \tag{76}$$

(b) 6 points

The initial value problem is

$$\begin{aligned} A_n'(t) &= -(n^2 \kappa + \sigma) A_n(t) + f_n \\ A_n(0) &= 0 \end{aligned} \tag{77}$$

Using an integrating factor, Green's functions, variation of parameters, the method of undetermined coefficients, Laplace transforms, or simply applying the general solution formula gives

$$A_n(t) = \frac{f_n}{n^2 \kappa + \sigma} \left(1 - e^{-(n^2 \kappa + \sigma) t}\right) \tag{78}$$

(c) 6 points

$$f(x) = \text{Cos}^2 x = \frac{1}{2} + \frac{1}{2} \text{Cos}(2x) = \frac{1}{4} e^{-2 i x} + \frac{1}{2} + \frac{1}{4} e^{2 i x} \tag{79}$$

This means we have only three non-zero f_n

$$\begin{aligned} f_{-2} &= 1/4 \\ f_0 &= 1/2 \\ f_2 &= 1/4 \end{aligned} \tag{80}$$

According to our solution formula in parts (a) and (b), we have

$$u(x, t) = \frac{1/4}{4\kappa + \sigma} (1 - e^{-(4\kappa+\sigma) t})\, e^{-2 i n x} + \frac{1}{2\sigma} (1 - e^{-\sigma t}) + \frac{1/4}{4\kappa + \sigma} (1 - e^{-(4\kappa+\sigma) t})\, e^{2 i x} \tag{81}$$

This can be simplified

$$u(x, t) = \frac{1/2}{4\kappa + \sigma} (1 - e^{-(4\kappa+\sigma) t})\, \text{Cos}(2 n x) + \frac{1}{2\sigma} (1 - e^{-\sigma t}) \tag{82}$$

(d) 6 points

When $\sigma \to 0$ the coefficient function $A_0(t)$ will have a removable singularity

$$A_0(t) = \frac{f_0}{\sigma} (1 - e^{-\sigma t}) \tag{83}$$

Method 1

Going back to part (b) and setting σ=0 gives

$$\begin{aligned} A_n'(t) &= -n^2 \kappa A_n(t) + f_n \\ A_n(0) &= 0 \end{aligned} \tag{84}$$

The solutions to this are

$$A_n(t) = \begin{cases} \frac{f_n}{n^2 \kappa} (1 - e^{-n^2 \kappa t}) & n \neq 0 \\ f_0\, t & n = 0 \end{cases} \tag{85}$$

So for the particular case of part (c) we find

$$u(x, t) = \frac{1}{8\kappa}(1 - e^{-4\kappa t})\,\text{Cos}(2 n x) + \frac{t}{2} \tag{86}$$

Method 2

We might instead just calculate the limit of our solution from part (b) as $\sigma \to 0$ for fixed t.

$$\begin{aligned} \lim_{\sigma\to 0} A_{n>0} &= \lim_{\sigma\to 0}\left(\frac{f_n}{n^2\,\kappa + \sigma}\left(1 - e^{-(n^2\,\kappa+\sigma)\,t}\right)\right) = \frac{f_n}{n^2\,\kappa}\left(1 - e^{-n^2\,\kappa t}\right) \\ \lim_{\sigma\to 0} A_0 &= \lim_{\sigma\to 0}\left(\frac{f_0}{\sigma}(1 - e^{-\sigma\,t})\right) \overset{\text{L'Hopital}}{=} \lim_{\sigma\to 0}(f_0\,t\,e^{-\sigma\,t}) = f_0\,t \end{aligned} \tag{87}$$

In the specific case of part (c) we'd have

$$\begin{aligned} &\lim_{\sigma\to 0}\left(\frac{1/2}{4\kappa + \sigma}(1 - e^{-(4\kappa+\sigma)\,t})\,\text{Cos}(2 n x)\right) = \frac{1}{8\kappa}(1 - e^{-4\kappa t})\,\text{Cos}(2 n x) \\ &\lim_{\sigma\to 0} A_0 = \frac{t}{2} \end{aligned} \tag{88}$$

Regardless of method, notice that with σ=0 in both the general and special case there is a linear growth term (unless $f_0 = 0$). Since σ=0 there is no dissipation of heat to your finger, so the temperature of the ring will increase without bound as time progresses.

Note(do not grade this part): examination of total heat content.

The total amount of heat in the ring is proportional to

$$H(t) = \int_0^{2\pi} u(x, t)\,dx \tag{89}$$

By integrating the PDE and initial condition we find

$$\begin{aligned} H'(t) &= -\sigma\,H(t) + \int_0^{2\pi} f(x)\,dx \\ H(0) &= 0 \end{aligned} \tag{90}$$

For σ>0 the solution to this is

$$H(t) = \frac{1 - e^{-\sigma\,t}}{\sigma}\int_0^{2\pi} f(x)\,dx \tag{91}$$

This remains bounded for all time, and as $t\to\infty$ an equilibrium between heating and dissipation is reached

$$H_{eq} = \frac{1}{\sigma}\int_0^{2\pi} f(x)\,dx \tag{92}$$

For σ=0, the solution is

$$H(t) = t\int_0^{2\pi} f(x)\,dx > 0 \text{ since } f(x) > 0 \text{ describes a heat source not a sink.} \tag{93}$$

And the total heat becomes unbounded due to lack of heat dissipation.

Problem 5 (6+5+5 points)

$$\begin{aligned}&T_t = \kappa T_{zz}\\&T(0, t) = \Delta T \operatorname{Sin}(\omega_0 t)\\&T(\infty, t) = 0\end{aligned} \tag{94}$$

(a) 6 points

By definition

$$\hat{T}(x, f) = \int_{-\infty}^{\infty} T(z, t)\, e^{-2\pi i f t}\, dt \tag{95}$$

Transforming the right side of the equation gives

$$\kappa \int_{-\infty}^{\infty} T_{zz}(z, t)\, e^{-2\pi i f t}\, dt = \kappa \hat{T}_{zz} \tag{96}$$

Transforming the left side using integration by parts

$$\int_{-\infty}^{\infty} T_t(z, t)\, e^{-2\pi i f t}\, dt = (e^{-2\pi i f t}\, T)_{t=\infty} - (e^{-2\pi i f t}\, T)_{t=-\infty} + 2\pi i f \int_{-\infty}^{\infty} T(z, t)\, e^{-2\pi i f t}\, dt \tag{97}$$

The boundary terms vanish if we assume that T vanishes at t=±∞. So the transformed equation is

$$2\pi i f \hat{T} = \kappa \hat{T}_{zz} \tag{98}$$

Where the left side comes from the formula for the transform of a derivative. Transforming the right boundary condition gives

$$\hat{T}(\infty, f) = 0 \tag{99}$$

Recall the delta representation

$$\delta(f - a) = \int_{-\infty}^{\infty} e^{-2\pi i (f-a) t}\, dt$$

Transforming the left boundary condition gives

$$\begin{aligned}&\hat{T}(0, f) = \Delta T \int_{-\infty}^{\infty} \operatorname{Sin}(\omega_0 t)\, e^{-2\pi i f t}\, dt =\\&\quad \frac{\Delta T}{2 i} \int_{-\infty}^{\infty} (e^{i \omega_0 t} - e^{-i \omega_0 t})\, e^{-2\pi i f t}\, dt = \frac{\Delta T}{2 i}\left(\delta\left(f - \frac{\omega_0}{2\pi}\right) - \delta\left(f + \frac{\omega_0}{2\pi}\right)\right)\end{aligned} \tag{100}$$

(b) 5 points

$$\begin{aligned}&2\pi i f \hat{T} = \kappa \hat{T}_{zz}\\&\hat{T}(\infty, f) = 0\\&\hat{T}(0, f) = \frac{\Delta T}{2 i}\left(\delta\left(f - \frac{\omega_0}{2\pi}\right) - \delta\left(f + \frac{\omega_0}{2\pi}\right)\right)\end{aligned} \tag{101}$$

The equation has constant coefficients, so we know it has solutions of the form

$$e^{r z} \tag{102}$$

We plug this in and solve for r

$$r = \begin{cases} \pm\left(\frac{1+i}{\sqrt{2}}\right)\sqrt{\frac{2\pi f}{\kappa}} & f > 0 \\ \pm\left(\frac{1-i}{\sqrt{2}}\right)\sqrt{\frac{-2\pi f}{\kappa}} & f < 0 \end{cases} \tag{103}$$

So solutions are of the form

$$\hat{T} = \begin{cases} A\, e^{\left(\frac{1+i}{\sqrt{2}}\right)\sqrt{\frac{2\pi f}{\kappa}}\, z} + B\, e^{-\left(\frac{1+i}{\sqrt{2}}\right)\sqrt{\frac{2\pi f}{\kappa}}\, z} & f > 0 \\ C\, e^{\left(\frac{1-i}{\sqrt{2}}\right)\sqrt{\frac{-2\pi f}{\kappa}}\, z} + D\, e^{-\left(\frac{1-i}{\sqrt{2}}\right)\sqrt{\frac{-2\pi f}{\kappa}}\, z} & f < 0 \end{cases} \tag{104}$$

Applying the boundary data gives

$$\hat{T} = \begin{cases} \frac{\Delta T}{2i}\left(\delta\left(f - \frac{\omega_0}{2\pi}\right) - \delta\left(f + \frac{\omega_0}{2\pi}\right)\right) e^{-\left(\frac{1+i}{\sqrt{2}}\right)\sqrt{\frac{2\pi f}{\kappa}}\, z} & f > 0 \\ \frac{\Delta T}{2i}\left(\delta\left(f - \frac{\omega_0}{2\pi}\right) - \delta\left(f + \frac{\omega_0}{2\pi}\right)\right) e^{-\left(\frac{1-i}{\sqrt{2}}\right)\sqrt{\frac{-2\pi f}{\kappa}}\, z} & f < 0 \end{cases} \tag{105}$$

Since ω_0>0 this can be more compactly written as

$$\hat{T} = \begin{cases} \frac{\Delta T}{2i}\,\delta\left(f - \frac{\omega_0}{2\pi}\right) e^{-\left(\frac{1+i}{\sqrt{2}}\right)\sqrt{\frac{2\pi f}{\kappa}}\, z} & f > 0 \\ -\frac{\Delta T}{2i}\,\delta\left(f + \frac{\omega_0}{2\pi}\right) e^{-\left(\frac{1-i}{\sqrt{2}}\right)\sqrt{\frac{-2\pi f}{\kappa}}\, z} & f < 0 \end{cases} \tag{106}$$

■ (c) 5 points

According to the definition of the inverse transform we have

$$T(z, t) = \int_{-\infty}^{\infty} \hat{T}(z, f)\, e^{2\pi i f t}\, df \tag{107}$$

Plugging in the expression from part (b) gives

$$T(z, t) = -\int_{-\infty}^{0} \frac{\Delta T}{2i}\,\delta\left(f + \frac{\omega_0}{2\pi}\right) e^{-\left(\frac{1-i}{\sqrt{2}}\right)\sqrt{\frac{-2\pi f}{\kappa}}\, z}\, e^{2\pi i f t}\, df + \int_{0}^{\infty} \frac{\Delta T}{2i}\,\delta\left(f - \frac{\omega_0}{2\pi}\right) e^{-\left(\frac{1+i}{\sqrt{2}}\right)\sqrt{\frac{2\pi f}{\kappa}}\, z}\, e^{2\pi i f t}\, df \tag{108}$$

This is simple to integrate because of the delta functions

$$T(z, t) = \frac{-\Delta T}{2i}\, e^{-\left(\frac{1-i}{\sqrt{2}}\right)\sqrt{\frac{\omega_0}{\kappa}}\, z}\, e^{-i\omega_0 t} + \frac{\Delta T}{2i}\, e^{-\left(\frac{1+i}{\sqrt{2}}\right)\sqrt{\frac{\omega_0}{\kappa}}\, z}\, e^{i\omega_0 t} \tag{109}$$

This simplifies to

$$T(x, t) = \Delta T\, e^{-z\sqrt{\frac{\omega_0}{2\kappa}}}\, \mathrm{Sin}\left(\omega_0 t - z\sqrt{\frac{\omega_0}{2\kappa}}\right) \tag{110}$$